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Covering group theory for topological groups

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Abstract

We develop a covering group theory for a large category of "coverable" topological groups, with a generalized notion of "cover". Coverable groups include, for example, all metrizable, connected, locally connected groups, and even many totally disconnected groups. Our covering group theory produces a categorial notion of fundamental group, which, in contrast to traditional theory, is naturally a (prodiscrete) topological group. Central to our work is a link between the fundamental group and global extension properties of local group homomorphisms. We provide methods for computing the fundamental group of inverse limits and dense subgroups or completions of coverable groups. Our theory includes as special cases the traditional theory of Poincaré, as well as alternative theories due to Chevalley, Tits, and Hoffmann–Morris. We include a number of examples and open problems. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction and main results

The purpose of this paper is to develop a covering group theory for a large category of (always Hausdorff!) topological groups. The traditional approach to this problem is to consider topological groups as topological spaces, and apply to them a theory developed in the purely topological setting. In this paper we consider only topological groups from the beginning, replacing traditional, purely topological assumptions by apparently more natural algebraic/topological conditions. The result is considerable simplification of proofs of traditional theorems, and significant generalization.

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For this theory we utilize a simple but natural construction discovered by Schreier in 1925 [22], rediscovered by Tits [24], and rediscovered in the more general setting of local groups by Mal'tsev [15]. In this construction a symmetric neighborhood U of the identity of a topological group G is isomorphically embedded in a uniquely determined topological group G_U that we call the Schreier group of G with respect to U (see Sections 2–5 of this paper for a more precise discussion of Schreier groups and the construction we now sketch). If one applies Schreier's construction to a symmetric neighborhood U of the identity e of a group G, there is an inclusion-induced open homomorphism $\phi_{GU}: G_U \to G$ with discrete kernel. If G is connected then ϕ_{GU} is surjective and hence, by definition, a traditional cover (although G_U may not be connected). For a connected, locally arcwise connected, locally simply connected group G, ϕ_{GU} is the universal cover of G when U is connected and small enough.

For an arbitrary topological group G, applying Schreier's construction to a pair $V \subset U$ of symmetric neighborhoods of e in G results in a homomorphism $\phi_{UV}: G_V \to G_U$ induced by the inclusion of U in V. This observation leads to an inverse system $\{G_U, \phi_{UV}\}$ indexed on the set of all symmetric neighborhoods of e in G, partially ordered by reverse inclusion. We denote the inverse limit of this system by $\widetilde{G}$. This inverse limit construction was first considered for metrizable groups by Kawada [14]. In [24], Tits independently considered an equivalent form of the same construction, and showed that if G has a simply connected traditional cover in the sense of Chevalley, then the natural projection $\phi: \widetilde{G} \to G$ must be that cover.

For any topological group G the kernel of the natural projection $\phi: \widetilde{G} \to G$ is central and prodiscrete (an inverse limit of discrete groups). In [4] we introduced the following generalized notion of cover of topological groups that we will use (for simplicity, in this paper homomorphisms are always continuous and we use the term "epimorphism" to mean "surjective homomorphism").

Definition 1. A homomorphism $\phi: G \to H$ between topological groups is called a cover if it is an open epimorphism with central, prodiscrete kernel.

In [4] we used "hemidiscrete" instead of "prodiscrete", which is used elsewhere in the literature. Because we considered connected groups in [4] we did not need the extra assumption of centrality. In fact, it is well known that any totally disconnected (hence any prodiscrete) normal subgroup of a connected group is central, but in general we do not know whether the requirement that the kernel be central can be removed (see Problem 134). A form of generalized cover was considered by Kawada, but his definition is flawed and his uniqueness theorem for generalized universal covers [14, Theorem 4] is incorrect; see Example 100. For Lie groups the notions of cover (in the present sense) and traditional cover are equivalent (see Section 7), but this is not true in general. The transition from discrete kernels to prodiscrete kernels allows us to completely eliminate from our theory any requirement of local simple connectivity in any form.

The principal problem now becomes:

Problem 2. For which categories $\mathcal{C}$ of topological groups do the following hold:

(1) For every $G \in \mathcal{C}$, $\phi : \widetilde{G} \to G$ is a cover.
(2) Covers are morphisms in $\mathcal{C}$ (i.e., the composition of covers between elements of $\mathcal{C}$ is a cover).
(3) The cover $\phi : \widetilde{G} \to G$ has the traditional universal property of the universal cover in the category $\mathcal{C}$ with covers as morphisms.

One of the main goals of this paper is to show that the above conditions are satisfied by a large category of topological groups, called coverable topological groups.

Definition 3. Let $\mathcal{C}$ be a category of topological groups. A topological group G is called locally defined (in $\mathcal{C}$) if there is a basis for the topology of G at e consisting of symmetric open sets U with the following extension property: for any $H \in \mathcal{C}$ and (local group) homomorphism $\psi : U \to H$, ψ extends uniquely to a homomorphism $\psi' : G \to H$. A group K is called coverable (in $\mathcal{C}$) if $K = G/H$ for some locally defined group G and closed normal subgroup H of G.

By a homomorphism between local groups U and V we simply mean a continuous function $\phi : U \to V$ such that whenever $a, b, ab \in U$, it follows that $\phi(a)\phi(b)$ lies in V and $\phi(ab) = \phi(a)\phi(b)$. The term "locally defined" refers to the easily proved fact that if two locally defined groups G and H have isomorphic symmetric neighborhoods of e then G and H are isomorphic. In the present paper we are mostly concerned with the category $\mathcal{T}$ of all topological groups. Normally we will omit mention of the specific category and the category is assumed to be $\mathcal{T}$; hence "coverable" means "coverable in $\mathcal{T}$". We consider the special case of locally compact groups (as elements of $\mathcal{T}$) in [2]. In [3] we consider the category $\mathcal{K}$ of compact, connected groups.

Chevalley considered extensions of local group homomorphisms and showed that a topological group that is connected, locally connected and "simply connected" in a certain sense, is satisfies our definition of locally defined [8, Theorem I.VII.3], cf. also Corollary 118 in the present paper. Since the universal covering group of a connected Lie group is simply connected, hence locally defined, every connected Lie group is coverable. However, the category of coverable groups is much larger, including, for example, all metrizable, connected, locally connected topological groups (Corollary 93) and even some totally disconnected groups (see Example 130). It follows easily from the definitions that the direct product of (arbitrarily many) locally defined groups is locally defined, and hence the direct product of coverable groups is coverable. Any quotient of a coverable group via a closed normal subgroup is clearly coverable; any dense subgroup of a coverable group, or the completion of any metrizable coverable group (if the completion is a group), is also coverable (Theorem 15).

From Proposition 78 and Theorem 90 we have:

Theorem 4. *If G is coverable then $\widetilde{G}$ is locally defined, and the natural homomorphism $\phi : \widetilde{G} \to G$ is a cover. If G is metrizable then $\widetilde{G}$ is metrizable.*

Theorem 5. *Let G_1, G_2 be coverable groups, and $\psi : G_1 \to G_2$ be a homomorphism. There exists a unique homomorphism $\widetilde{\psi} : \widetilde{G}_1 \to \widetilde{G}_2$ such that, if $\phi_1 : \widetilde{G}_1 \to G_1$, $\phi_2 : \widetilde{G}_2 \to G_2$ denote the respective natural homomorphisms, then $\phi_2 \circ \widetilde{\psi} = \psi \circ \phi_1$.*

Theorem 6. *If $\psi : G_1 \to G_2$ and $\pi : G_2 \to G_3$ are covers between coverable groups then $\pi \circ \psi$ is a cover.*

Proving that covers are closed under composition does not seem to be an easy task in general, and our proof for coverable groups requires a preliminary version of the universal property of $\widetilde{G}$ (Theorem 101); see Problem 135. Theorem 6 implies that coverable groups, with covers as morphisms, form a category, and the following is the traditional universal property of universal covering homomorphisms in this category.

Theorem 7. *Let $\pi : G \to H$ be a cover between coverable groups G and H. Then there is a unique cover $\psi : \widetilde{H} \to G$ such that $\phi = \pi \circ \psi$, where $\phi : \widetilde{H} \to H$ is the natural epimorphism.*

Given the above theorem, if G is coverable then we are justified in calling $\widetilde{G}$ the *universal covering group* of G and ϕ the *universal covering epimorphism* of G. The standard arguments imply that $\widetilde{G}$ is the unique (up to isomorphism) group in this category having the universal property stated in Theorem 7.

The central (hence Abelian), prodiscrete subgroup $K := \ker \phi$ of $\widetilde{G}$ can be identified with the traditional (Poincaré) fundamental group of G in many natural circumstances (see Sections 5 and 7), including when G is connected, locally arcwise connected, and semilocally simply connected; in that case $\widetilde{G}$ is the universal cover of G in the traditional sense. We therefore denote K by $\pi_1(G)$ and call it the fundamental group of G. If G_1 and G_2 are coverable, and $\psi : G_1 \to G_2$ is a homomorphism, then $\widetilde{\psi}$ (cf. Theorem 5) restricted to $\pi_1(G_1)$ is a homomorphism into $\pi_1(G_2)$. We denote by $\psi_* : \pi_1(G_1) \to \pi_1(G_2)$ this restriction, and refer to it as the induced homomorphism of the fundamental group. Clearly it is functorial. Note that our fundamental group is in fact a topological group, and the induced homomorphism is a continuous homomorphism.

Theorem 8. *Let G_1, G_2, G_3 be coverable groups, $\psi : G_1 \to G_2$ be a homomorphism and $\pi : G_3 \to G_2$ be a cover. Then $\psi_*(\pi_1(G_1)) \subset \pi_*(\pi_1(G_3))$ if and only if there is a homomorphism $\psi' : G_1 \to G_3$ such that $\pi \circ \psi' = \psi$. If ψ' exists, it is unique. Moreover, if ψ is a cover then ψ' is a cover. In this case ψ' is an isomorphism if and only if $\psi_*(\pi_1(G_1)) = \pi_*(\pi_1(G_3))$.*

Theorem 9. *Let G and H be coverable groups. If $\pi : G \to H$ is a cover then $\pi_*(\pi_1(G))$ is a closed subgroup of $\pi_1(H)$. Given any closed subgroup K of $\pi_1(G)$ there is a unique (up to isomorphism of covers) cover $\pi : G' \to G$, for some coverable group G', such that $\pi_*(\pi_1(G')) = K$.*

In the next proposition, sufficiency follows from the fact that $\widetilde{G}$ is locally defined when G is coverable (Theorem 90), and necessity follows from Corollary 71.

Proposition 10. *A coverable group G is locally defined if and only if $\pi_1(G)$ is trivial.*

From the above proposition and Theorem 8 we obtain that any locally defined covering group must be the universal covering group:

Corollary 11. *Let $\pi : H \to G$ be a cover between topological groups G and H. If H is locally defined then there is a unique isomorphism $\psi : \widetilde{G} \to H$ such that $\phi = \pi \circ \psi$, where $\phi : \widetilde{G} \to G$ is the universal covering epimorphism.*

We now state some results useful for computing fundamental groups. Given a homomorphism $\psi : G \to H$ between topological groups, there is a natural induced homomorphism $\widetilde{\psi} : \widetilde{G} \to \widetilde{H}$ satisfying a natural, but somewhat complicated, uniqueness property (Theorem 73). By uniquess, if G and H are coverable, $\widetilde{\psi}$ coincides with the homomorphism given Theorem 5. We can now state the following:

Theorem 12. *Let $(G_\alpha, p_{\alpha\beta})$ be an inverse system of topological groups with inverse limit G such that the bonding homomorphisms $p_{\alpha\beta}$ are open and the natural homomorphisms $p_\alpha : G \to G_\alpha$ are surjective. Then $G' := \varprojlim(\widetilde{G}_\alpha, \widetilde{p}_{\alpha\beta})$ is naturally isomorphic to $\widetilde{G}$.*

Remark 13. If the groups G_α in Theorem 12 are locally defined, then G is locally defined (Corollary 68), but as Example 99 shows, if the groups G_α are coverable, G need not be coverable. If each of the groups G_α is generated by each neighborhood of the identity, in particular if each G_α is connected or coverable, then the open bonding homomorphisms must be surjective (see Section 2). By Lemmas 39 and 40, if in addition the above inverse system has a countable indexing set then we need not assume the homomorphisms p_α are surjections.

Corollary 14. *For any collection $\{G_\alpha\}$ of topological groups, $\widetilde{\prod G_\alpha}$ is naturally isomorphic to $\prod \widetilde{G}_\alpha$, where "$\prod$" denotes the direct product.*

Theorem 15. *Let H be a dense subgroup of a topological group G. If G is coverable then H is coverable. If H is coverable and either G is metrizable or $\phi : \widetilde{G} \to G$ is surjective, then G is coverable. If both H and G are coverable and i denotes the inclusion, then the homomorphism $\tilde{i} : \widetilde{H} \to \widetilde{G}$ is an isomorphism onto a dense subgroup of $\widetilde{G}$ and the induced homomorphism $i_* : \pi_1(H) \to \pi_1(G)$ is an isomorphism.*

Note that, in traditional fundamental group theory, the inclusion of a dense subgroup into topological group need not induce an isomorphism of the fundamental group.

The organization of this paper is as follows. In the next three sections we lay the groundwork for our paper, including characterizations of Schreier groups and their extension properties, as well as a few preliminaries about inverse limits. Section 5 is concerned with the construction of $\widetilde{G}$ and its properties, including the relationship between $\ker\phi$ and the traditional (Poincaré) fundamental group. In Section 6 we study coverable groups, proving, in addition to the main theorems mentioned above, a useful intrinsic characterization of coverable groups (Theorem 90). In Section 7 we study traditional covers, giving a new theory that extends the work of Tits and subsumes the work of Poincaré, Chevalley, and Hofmann–Morris, while fitting nicely into our more general framework. In Section 8 we consider various special cases, and in the last section we give a list of open problems. Examples are included throughout the paper.

We would like to add here some discussion suggested by the referee. First, a number of results about Schreier groups are also true at the purely algebraic level. For example, if one considers homomorphisms only in the algebraic sense and open neighborhoods only as sets, then purely algebraic analogs of Propositions 53 through 60 and Lemma 64 through Proposition 66 are valid. In addition, more extensive use of category theory would allow more formal statements and sometimes shorter proofs of some results in this paper. However, not being ourselves experts in category theory (and hoping that other non-experts will be interested in our work), we elected to not expand our use of category theory. We also wonder whether the neat machinery of category theory might not hide the essentially geometric nature of our work, making it harder to even imagine (much less prove) results like Theorem 15. For the benefit of category theory experts, we provide here the referee's translation of some results of Section 6, where $\mathcal{C}$ and $\mathcal{S}$ denote the full subcategories of $\mathcal{T}$ of coverable and locally defined groups, respectively:

Theorem 16. *The self-functor $\tilde{}$ of $\mathcal{T}$ induces a functor $\tilde{}:\mathcal{C}\to\mathcal{S}$ which is right adjoint to the forgetful functor and $\phi:\widetilde{G}\to G$ is the counit of the adjunction. The counit is a cover whose* (*prodiscrete*) *kernel is denoted $\pi_1(G)$.*

2. Locally generated and prodiscrete groups

Definition 17. A topological group G is called locally generated if it is generated by each neighborhood of e.

Remark 18. The natural question of whether complete locally generated groups must be connected (the rationals numbers are locally generated but not complete) was asked more than 60 years ago by Mazur [16, Problem 160], cf. also [9, p. 103], and answered in the negative by Stevens [23].

Definition 19. Let G be a group and U an open neighborhood of e in G. A U-chain from e to $x\in G$ is a finite sequence $\{x_0=e,x_1,\dots,x_n=x\}$ of elements of G such that $x_i^{-1}x_{i+1}\in U$ for all i. A G-chain will simply be referred to as a chain. If $\phi:G\to H$ is a homomorphism and $c=\{x_0,x_1,\dots,x_n\}$ is a chain in G, then by $\phi(c)$ we mean the chain $\{\phi(x_0),\phi(x_1),\dots,\phi(x_n)\}$ in H.

Proposition 20. *The following are equivalent for a topological group G:*

(1) *G is locally generated.*
(2) *G contains no proper open subgroup.*
(3) *For any $x \in G$ and neighborhood U of e, there is a U-chain from e to x.*

Proof. If a group contains a proper open subgroup H, then H cannot generate G. Conversely, suppose G has an open neighborhood U of the identity that does not generate it. Then the subgroup generated by U, being a union of open sets, must be a proper open subgroup of G. We have proved the equivalence of the first two conditions.

G is locally generated if and only if for any neighborhood U of e and every $g \in G$, there exist $g_1, \ldots, g_n \in U$ such that $g = g_1 \cdots g_n$. Letting $x_i := g_1 \cdots g_i$ we see that $\{x_0 := e, x_1, \ldots, x_n = g\}$ is precisely a U-chain to g. Conversely, given any U-chain $\{e, x_1, \ldots, x_n\}$ we can set $g_i = x_{i-1}^{-1} x_i$ to verify that G is locally generated, proving the equivalence of (1) and (3). □

If a topological group G has a connected neighborhood U of e, then the subgroup of G generated by U is a connected open subgroup of G. We obtain:

Corollary 21. *If G is a locally generated group then G is connected if and only if G is has a connected neighborhood of e.*

Corollary 22. *If G is a locally connected topological group then G is connected if and only if G is locally generated.*

Lemma 23. *Let G be a group and H be a locally generated subgroup. Then the closure $\overline{H}$ of H in G is locally generated.*

Proof. Let V be a neighborhood of e in G. Then $V \cap \overline{H}$ generates an open, hence closed, subgroup K of $\overline{H}$. But since $H \cap V$ is contained in $\overline{H} \cap V$ and generates H, we have $H \subset K \subset \overline{H}$, and therefore $K = \overline{H}$. □

Corollary 24. *The completion of a locally generated group* (*if it is a group*) *is locally generated.*

Proposition 25. *If H is a dense subgroup of a topological group G then H is locally generated if and only if G is locally generated.*

Proof. Let H be a dense subgroup of a locally generated group G, U be an open neighborhood of e in G, and h be an element of H. Let $\{x_0, \ldots, x_n\}$ be a U-chain from e to h in G. By the continuity of the product and the fact that H is dense, there exist $y_1, \ldots, y_{n-1} \in H$ so that y_i is close enough to x_i that (also setting $y_0 := e$ and $y_n = h$), $y_i^{-1} y_{i+1}$ is also in U. So $\{y_0, \ldots, y_n\}$ is a U-chain to h in H.

The converse is immediate from Lemma 23. □

Note that a statement similar to the above proposition may be found in [11]. The proof of the next lemma is obvious.

Lemma 26. *If $\phi : G \to H$ is an epimorphism and G is locally generated then H is locally generated.*

Lemma 27. *Let H be locally generated. If $\phi : G \to H$ is an open homomorphism then ϕ is surjective.*

Proof. Since ϕ is open, $\phi(G)$ is an open subgroup of H, and surjectivity follows from Proposition 20. □

Lemma 28. *Let G be a topological group such that for every open set U containing e there exists a closed normal subgroup $H \subset U$ such that G/H is generated by $\pi(U)$, where $\pi : G \to G/H$ is the quotient homomorphism. Then G is locally generated.*

Proof. Suppose G is not locally generated, and let K be a proper open subgroup of G; i.e., there exists $x \in G \backslash K$. Let $H \subset K$ be a closed subgroup, normal in G, such that G/H is generated by $\pi(K)$. Since $\pi(K)$ is a subgroup of G/H, $\pi(K) = G/H$. Therefore, there must be some $y \in K$ such that $\pi(y) = \pi(x)$. In other words, $x^{-1}y \in H \subset K$. But then $y \in xK$, which contradicts the fact that the cosets xK and K are disjoint. □

The following proposition establishes the analog of the connected component. Recall that a subgroup K of a group H is *characteristic* if every automorphism of H takes K onto itself (some authors only require that K be taken into itself). We say K is *fully characteristic* if each endomorphism of H restricts to an endomorphism of K. Note that if K is fully characteristic in H then K is normal in H. The *quasicomponent* of a topological group G is the intersection of all open subgroups of G.

Proposition 29. *Let G be a topological group. Then G contains a largest locally generated subgroup G^l (i.e., G^l is locally generated, and contains every locally generated subgroup of G). G^l is closed, fully characteristic (hence normal), contains the identity component of G, and is contained in the quasicomponent of G.*

Proof. First note that the connected component of G (being connected) is locally generated. Let G^l be the subgroup of G generated by the union $\mathcal{U}$ of all locally generated subgroups of G. By Lemma 23 we need only show that G^l is locally generated. Let $V \ni e$ be an open subset of G, and let $x \in G^l$. Then $x = x_1 \cdots x_n$, where $x_i \in H_i$ for some locally generated subgroups H_i of G. But then, $x_i = y_{i1} \cdots y_{ik(i)}$, where each $y_{ij} \in H_i \cap V \subset G^l \cap V$, so G^l is locally generated. If $h : G \to G$ is an endomorphism of G then $h(G^l)$ is another locally generated subgroup of G by Lemma 26. By the maximality of G^l, $h(G^l) \subset G^l$; i.e., G^l is fully characteristic. Finally, suppose K is an open subgroup of G. Let V be any neighborhood of e in K. Then since G^l is generated by $V \cap G^l$, G^l is contained in the group generated by V, which in turn is contained in K. □

Definition 30. The subgroup G^l is called the l-component of G.

If G is locally compact, then the quasicomponent of G is equal to the identity component of G (cf. [6, p. 260]); in particular, the l-component is the identity component by Proposition 29. We obtain:

Proposition 31. *If G is locally compact then G is locally generated if and only if G is connected.*

The following lemma follows from [6, III.7.3, Proposition 2].

Lemma 32. *A topological group G is prodiscrete if and only if G is complete and every neighborhood of e contains an open normal subgroup.*

A corollary of Lemma 32 is that prodiscrete groups are totally disconnected. However, there are (even locally compact) totally disconnected groups that are not prodiscrete [18]. The proof of the next lemma can be found in [4]:

Lemma 33.
(1) *Any closed subgroup of a prodiscrete group is prodiscrete.*
(2) *If G is prodiscrete and H is a closed normal subgroup of G then G/H is prodiscrete.*
(3) *If G is the direct product* (*possibly infinite*) *or inverse limit of prodiscrete groups then G is prodiscrete.*

It is well known and easy to prove (cf. [18]) that any totally disconnected normal subgroup of a connected group must be central. However, this result fails for locally generated groups in general. The following example was suggested by the referee:

Example 34. Let

$$L = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} :\ 0 < a \in \mathbb{R},\ b \in \mathbb{R} \right\},$$

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} :\ 0 < a \in \mathbb{Q},\ b \in \mathbb{Q} \right\},$$

and

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} :\ b \in \mathbb{R} \right\}.$$

Then L is a connected Lie group, hence coverable, and centerfree. G is a dense, totally disconnected subgroup of L, hence coverable by Theorem 15, and centerfree. Now N is normal in L, so $N \cap G$ is a normal totally disconnected subgroup of the coverable group G, but is not central.

Of course $N \cap G$ is not prodiscrete in the above example, but we do not know whether every prodiscrete normal subgroup of a locally generated group is central either (see Problem 134). Nonetheless we can manage with the following lemma.

Lemma 35. *Suppose G is locally generated. Let H be a closed normal subgroup of G such that for every neighborhood U of e in G there exists an open subgroup K of H, contained in U and normal in G. Then H is central in G. If in addition H is complete, H is prodiscrete.*

Proof. Suppose that for some $x \in G$, $y \in H$, $x^{-1}yx = z \neq y$. Let V be open in G about e such that $y^{-1}z \notin V$, let $K = H \cap U$ be an open subgroup of H, normal in G where the open set U of G is contained in V. Finally, let W be a neighborhood of e in G so that for all $w \in W$, $y^{-1}w^{-1}yw \in U$—which implies, since H is normal, $y^{-1}w^{-1}yw \in K$, so $w^{-1}yw \in yK$. Let $\pi : H \to H/K$ denote the quotient epimorphism to the discrete group $D := H/K$. Then G acts continuously on D via the automorphisms $\phi_w : D \to D$ given by $\phi_w(\pi(a)) = \pi(w^{-1}aw)$, for any $w \in G$. Then ϕ_w is well defined because K is normal in G. If $w \in W$, $\pi(w^{-1}yw) = \pi(y)$, so $\phi_w(\pi(y)) = \pi(y)$. Writing $x = x_1 \cdots x_n$, where $x_i \in W$, each ϕ_{x_i} fixes $\pi(y)$, and we see that

$$\pi(z) = \pi\big(x^{-1}yx\big) = \phi_x\big(\pi(y)\big) = \phi_{x_n} \circ \cdots \circ \phi_{x_1}\big(\pi(y)\big) = \pi(y).$$

That is, $y^{-1}z \in K \subset V$, a contradiction. The last statement of the lemma follows from Lemma 32. □

Corollary 36. *If H is a discrete normal subgroup of a locally generated group G then H is central.*

3. Preliminaries on inverse limits

For the basic definitions and results about inverse limits, see [6] or [13]. We give here a few basic results we need; we prove those for which we have no references. For this section we fix an inverse system $\{G_\alpha, \pi_{\alpha\beta}\}$ of topological groups and bonding homomorphisms $\pi_{\alpha\beta} : G_\beta \to G_\alpha$ ($\alpha \leqslant \beta$). By definition, the indexing set is a partially ordered set that is also directed, and the bonding homomorphisms satisfy $\pi_{\alpha\beta} = \pi_{\alpha\gamma} \circ \pi_{\gamma\beta}$ whenever $\alpha \leqslant \gamma \leqslant \beta$. The inverse limit of this system is $G = \{(x_\alpha)\colon\ x_\alpha \in G_\alpha$ and $x_\alpha = \pi_{\alpha\beta}(x_\beta)$, whenever $\alpha \leqslant \beta\}$. We denote by $\pi_\alpha : G \to G_\alpha$ the restriction of the coordinate projection homomorphism defined for ΠG_α. The group G has the following *universal property*: Given any topological group H and collection of homomorphisms $\{\phi_\alpha : H \to G_\alpha\}$ such that for all $\beta \geqslant \alpha$, $\phi_\alpha = \phi_{\alpha\beta} \circ \phi_\beta$ there exists a unique homomorphism $\phi : H \to G$ such that $\phi_\alpha = \pi_\alpha \circ \phi$ for all α. A proof of the next lemma may be found in [6].

Lemma 37. *Let β be a fixed index. A basis for the topology of $G = \varprojlim G_\alpha \subset \Pi G_\alpha$ consists of all sets of the form $\pi_\alpha^{-1}(U)$, where U is open in G_α and $\alpha \geqslant \beta$.*

The next result is well known and easy to prove. We will use it frequently without reference.

Proposition 38. *Let $\{G_{\alpha_\gamma}\}$ be a subcollection of $\{G_\alpha\}$ such that for every α there exists a γ such that $\alpha_\gamma \geqslant \alpha$. Then there is a natural isomorphism $i : \varprojlim G_{\alpha_\gamma} \to \varprojlim G_\alpha$.*

Lemma 39. *If the indexing set is countable and the bonding homomorphisms $\pi_{\alpha\beta}$ are surjective, then the homomorphisms π_α are surjective.*

Proof. By Proposition 38 we can suppose the system is indexed using integers. Fix $x_i \in G_i$. For all $k \leqslant i$, let $x_k = \pi_{ki}(x_i)$. We can iteratively choose a sequence $\{x_j\}$ in the following way. We have already chosen x_j for all $j \leqslant i$. Suppose we have chosen x_j for $j < m$. Since $\pi_{(m-1)m}$ is surjective we can choose x_m such that $\pi_{(m-1)m}(x_m) = x_{m-1}$. The sequence constructed in this way is clearly in G, and $\pi_i((x_j)) = x_i$. $\square$

Lemma 40. *If the bonding homomorphisms $\pi_{\alpha\beta}$ are open and the homomorphisms $\pi_\alpha : G \to G_\alpha$ are surjective, then the homomorphisms π_α are open.*

Proof. Consider a basis element $V := \pi_\alpha^{-1}(U)$, where U is open in G_α. If $\beta \geqslant \alpha$, then since π_β is surjective, $\pi_\beta(V) = \pi_\beta(\pi_\beta^{-1}(\pi_{\alpha\beta}^{-1}(U))) = \pi_{\alpha\beta}^{-1}(U)$ for all $\beta \geqslant \alpha$; i.e., π_β is open. Similarly, if $\gamma \leqslant \alpha$, then since π_α is surjective,

$$\pi_\gamma(V) = \pi_{\gamma\alpha}\big(\pi_\alpha(V)\big) = \pi_{\gamma\alpha}\big(\pi_\alpha\big(\pi_\alpha^{-1}(U)\big)\big) = \pi_{\gamma\alpha}(U),$$

which is open in G_γ, since $\pi_{\gamma\alpha}$ is open. $\square$

From the previous lemma and Lemma 28 we obtain:

Lemma 41. *If each G_α is locally generated, the bonding homomorphisms $\pi_{\alpha\beta}$ are open and the homomorphisms $\pi_\alpha : G \to G_\alpha$ are surjective, then G is locally generated.*

Lemma 42. *If each G_α is locally generated, the bonding homomorphisms $\pi_{\alpha\beta}$ are open with discrete kernel, and there is some α_0 such that $\pi_{\alpha_0} : G \to G_{\alpha_0}$ is surjective then all the homomorphisms π_α are open epimorphisms and G is locally generated.*

Proof. Since each G_α is locally generated, by Lemma 27, each $\pi_{\alpha\beta}$ is surjective. Then for any $\alpha \leqslant \alpha_0$, we have $\pi_\alpha(G) = \pi_{\alpha\alpha_0}(\pi_{\alpha_0}(G)) = G_{\alpha_0}$, i.e., π_α is an epimorphism. Now suppose that $\alpha_0 \leqslant \alpha$. Since $\pi_{\alpha_0\alpha}$ is a traditional cover there is an open neighborhood U_{α_0} of e in G_{α_0} such that $\pi_{\alpha_0\alpha}^{-1}(U_{\alpha_0})$ is a disjoint union of open sets $U_{\alpha\gamma}$, where $U_{\alpha\gamma} := x_\gamma U_\alpha$, with $x_\gamma \in K_{\alpha_0\alpha} := \ker \phi_{\alpha_0\alpha}$ and $e \in U_\alpha$. Then the restriction of $\pi_{\alpha_0\alpha}$ to each $U_{\alpha\gamma}$ is a homeomorphism. Now let $U := \pi_{\alpha_0}^{-1}(U_{\alpha_0})$. Then for $K_{\alpha_0} := \ker \pi_{\alpha_0}$, $K'_{\alpha_0\alpha} := \pi_\alpha(K_{\alpha_0})$ is a subgroup of the discrete kernel $K_{\alpha_0\alpha}$. The latter is central in G_α by Lemma 35, so $K'_{\alpha_0\alpha}$ is also a central discrete subgroup of G_α. Evidently we have

$$U = K_{\alpha_0}U \quad \text{and} \quad \pi_\alpha(U) = \pi_\alpha(K_{\alpha_0}U) = K'_{\alpha_0\alpha} \cdot \pi_\alpha(U) \subset K_{\alpha_0\alpha}U_\alpha.$$

Since π_α is a homomorphism and $K'_{\alpha_0\alpha} \subset \pi_\alpha(U)$, it follows that for $U'_\alpha := \pi_\alpha(U) \cap U_\alpha$, we must have $\pi_\alpha(U) = K'_{\alpha_0\alpha} U'_\alpha$. Then since $K'_{\alpha_0\alpha} \subset K_{\alpha_0\alpha}$, we have

$$U_{\alpha_0} = \pi_{\alpha_0}(U) = \pi_{\alpha_0\alpha}\big(\pi_\alpha(U)\big) = \pi_{\alpha_0\alpha}\big(K_{\alpha_0\alpha} U'_\alpha\big) = \pi_{\alpha_0\alpha}\big(U'_\alpha\big).$$

From this follows that $U'_\alpha = U_\alpha$, because $\pi_{\alpha_0\alpha} : U_\alpha \to U_{\alpha_0}$ is a homeomorphism. Hence $U_\alpha \subset \pi_\alpha(U) \subset \pi_\alpha(G)$ and π_α is surjective since G_α is generated by U_α and π_α is a homomorphism. Now if β is any index, then there exists an index δ such that $\delta \geqslant \beta$ and $\delta \geqslant \alpha_0$. By what we have just shown, π_δ, hence π_β, must be an epimorphism. By Lemma 40, each π_α must be an open epimorphism, and from Lemma 28, it follows that G is locally generated. $\square$

4. Schreier groups

For this section we will need a suitable definition of isomorphism of local groups. Here one must be careful. An isomorphism is defined to be a one-to-one and onto open homomorphism whose inverse is also a homomorphism. For example, if $U = \{e^{it}:\ t \in (-\pi, \pi)\}$, then the correspondence $t \to e^{it}$ is a (local group) homomorphism and homeomorphism that is *not* an isomorphism. A local isomorphism of a local group (or group) is an open homomorphism that is an isomorphism onto its image when restricted to some neighborhood of e. It is an easy exercise to show that if $\phi : U \to V$ is a local group homomorphism and homeomorphism, and $W \subset U$ is a symmetric neighborhood of e such that $W^2 \subset U$ then the restriction of ϕ to W is a local group isomorphism. In particular, if $\phi : G \to H$ is a homomorphism then ϕ is a local isomorphism $\Leftrightarrow$ ϕ is a local homeomorphism $\Leftrightarrow$ ϕ is open and has discrete kernel.

For our description of Schreier groups we follow Mal'tsev [15], in which the construction is considered for local groups or pseudogroups that are "associative" in a generalized sense that is always satisfied by symmetric neighborhoods of the identity in a topological group. Let G be a topological group and U be a symmetric neighborhood of e in G. Let $\mathcal{G}$ denote the semigroup of all words $a_1 \cdots a_n$ whose letters a_i are elements of U, $n = 1, 2, \ldots,$ where the product operation is concatenation of words (e.g., $abc \cdot def = abcdef$). There are two basic operations that can be performed on a word $a_1 \cdots a_n$. If the product c of two adjacent elements $a_i a_{i+1}$ lies in U, the word can be *contracted* by replacing $a_i a_{i+1}$ with c. The word $a_1 \cdots a_n$ can be *expanded* if some element $a_k = bc$, where $b, c \in U$, by replacing a_k with bc. Define an equivalence relation on $\mathcal{G}$ as follows. We say $a_1 \cdots a_n \equiv b_1 \cdots b_k$ if and only if $a_1 \cdots a_n$ can be transformed into $b_1 \cdots b_k$ by a finite number of expansions or contractions. We denote the equivalence class of $a_1 \cdots a_n$ by $[a_1 \cdots a_n]$. It is not hard to verify that the quotient space $G_U := \mathcal{G}/\equiv$ is a group with the operation induced by the semigroup operation. The mapping which sends each $a \in U$ to the equivalence class $[a]$ is a one-to-one function of U into G_U. We will often identify U with its image in G_U, and refer to the mapping $a \mapsto [a]$ as the "inclusion". Since G_U is generated by U, there exists a unique topology on G_U such that the inclusion of U in G_U is a homeomorphism onto an open set in G_U (see [24,15], or [12, Theorem A2.25] for more details). With this topology, the inclusion is a homomorphism from the local group U into G_U.

Definition 43. If G is a topological group and U is a symmetric neighborhood of e in G, the group G_U defined above will be called the *Schreier group* of G with respect to U.

This construction was introduced in [15] in order to prove that a local group can be embedded in a topological group if and only if the associative law is valid for products of arbitrary length. This "generalized associative law" always holds in a symmetric neighborhood of e in a group but may not be valid even locally in a more general local group. Tits later gave a different construction of G_U [24]. The next lemma is essentially proved in [15] or [24], but due to differences in definitions and notation, we give a proof here.

Lemma 44. *If G is a topological group then the inclusion homomorphism of U into G_U is a* (*local group*) *isomorphism onto its image. That is, we can identify U as a local group with its image in G_U.*

Proof. We already know from the construction that the inclusion is a one-to-one homomorphism, and have defined the topology on G_U to make it a homeomorphism onto its (open) image. We need to show that the function $[a] \to a$ is a homomorphism; that is, if $[a][b] = [c]$ then we need to show that $ab = c$ in G, and since $c \in U$, we are finished. First, note that $[ab] = [a][b] = [c]$ implies that the word ab can be transformed into the word c by some sequence of expansions or contractions. But expansions and contractions preserve the product (in G) of the elements of the word; hence $ab = c$. □

We will use the above lemma frequently without reference. Note that any local property of a topological group—such as first countability or completeness—is passed on to the Schreier group. Clearly $G_G \equiv G$.

We now present a useful alternative construction of G_U (cf. also [14]).

Definition 45. Let G be a topological group and U be a symmetric neighborhood of e. A U-extension of a U-chain $\{x_0, \ldots, x_n\}$ (see Definition 19) to $x := x_n$ is a U-chain $\{x_0, \ldots, x_i, x', x_{i+1}, \ldots, x_n\}$, where $0 < i < n$. Two U-chains are said to be U-related if one is a U-extension of the other. A U-homotopy between U-chains γ_0 and γ_m is a sequence $\{\gamma_0, \ldots, \gamma_m\}$ of U-chains such that γ_i is U-related to γ_{i-1} for all $1 \leqslant i \leqslant m$. We denote the U-homotopy class of a U-chain γ by $[\gamma]_U$. If $\phi : G \to H$ is a homomorphism and $h := \{\gamma_0, \ldots, \gamma_m\}$ is a U-homotopy, then by $\phi(h)$ we mean the H-homotopy $\{\phi(\gamma_0), \ldots, \phi(\gamma_m)\}$ in H.

Note that if $V \subset U$ then a V-chain is also a U-chain. Let $\gamma = \{x_0, \ldots, x_n\}$ be a U-chain. If U is symmetric then there corresponds to γ an element $\overline{\gamma} \in G_U$, namely $[a_1 \cdots a_n]$, where $a_i = x_{i-1}^{-1} x_i$. The proof of the next lemma is immediate from the definitions:

Lemma 46. *Two U-chains γ_1, γ_2 in a topological group G are U-homotopic if and only if $\overline{\gamma}_1 = \overline{\gamma}_2$ in G_U. Therefore the correspondence $[\gamma]_U \leftrightarrow \overline{\gamma}$ of U-homotopy classes of U-chains with G_U is bijective. If $a \in G_U$ then every U-chain corresponding to a is a U-chain in G to $\phi_{GU}(a)$. In this correspondence, the product $\overline{\gamma}_1\overline{\gamma}_2$ of elements $\overline{\gamma}_1, \overline{\gamma}_2 \in G_U$ corresponds to the U-equivalence class of the chain $\{x_0, \ldots, x_n, y_0, \ldots, y_m\}$, where $\gamma_1 = \{x_0, \ldots, x_n\}$ and $\gamma_2 = \{y_0, \ldots, y_m\}$.*

Remark 47. In studying a Schreier group G_U it is very easy to make mistakes by forgetting that Schreier's equivalence relation requires a sequence of binary operations, and that one is not allowed to make replacements involving products of more than two elements.

Example 48. In the additive group of the real numbers $\mathbb{R}$, let $U = (-1, 1)$, $V = (-4, -2) \cup (2, 4)$ and $W = U \cup V$. Note that a U-chain is a chain such that adjacent elements are of distance less than 1. A W-chain is a chain such that adjacent elements are of distance less than 1 or between 2 and 4. We will see in Theorem 113 that $\mathbb{R}_U \equiv \mathbb{R}$, and one can also verify this directly. According to [27], $\mathbb{R}_W$ is isomorphic to $\mathbb{Z} \times \mathbb{R}$, and more generally, if W is a union of finitely many intervals that are "independent" in a certain sense, $\mathbb{R}_W$ is isomorphic to the direct product of a finitely generated free group with $\mathbb{R}$. Note that the inclusion of W into $\mathbb{R}_W$ is a local group isomorphism that has no extension to $\mathbb{R}$. Finally, note that the disconnectedness of $\mathbb{R}_W$ is not simply a consequence of the topological fact that W is not connected! For example, if we instead let $V = (-2, -1) \cup (1, 2)$, then W is still not connected but $\mathbb{R}_W \equiv \mathbb{R}$.

Example 49. Let $G = S^1$. If $V := \{e^{it}\colon t \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)\}$ is a (multiplicative) local group then V is isomorphic to $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ and so G_V is isomorphic to $\mathbb{R}_{(-\pi/2,\pi/2)} = \mathbb{R}$. Suppose, on the other hand, we let $U := \{e^{it}\colon t \in (-\pi, \pi)\}$. Then a U-chain is a chain such that no two adjacent elements are antipodal. We will show that the natural homomorphism $\phi_{GU}: G_U \to G$ is an isomorphism. Since G is connected, ϕ_{GU} is surjective. To show ϕ_{GU} is injective, consider a U-loop $c := \{x_0 = 1, x_1, \ldots, x_n = 1\}$, where $n \geqslant 2$. We will be finished by induction if we can prove that c is U-homotopic to a shorter U-loop. If x_{i-1}, x_{i+1} are not antipodal for some i, then we can remove x_i to obtain U-chain shorter than c that is U-related (hence U-homotopic) to c. Suppose x_{i-1}, x_{i+1} are antipodal for all $i \in \{1, \ldots, n-1\}$. Then c is of the form $\{1, a, -1, a', 1, \ldots, 1\}$ where a and a' are antipodal. Now add a point b between -1 and a, then remove both -1 and a to complete the proof.

Example 50. Let $G = S^1 \times \{1, -1\}$ and let $U = \{(1, e^{it})\colon t \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)\}$ and $V = U \cup \{(-1, e^{it})\colon t \in (\frac{1}{2}\pi, \frac{3}{2}\pi)\}$. Then as we have seen above, G_U is isomorphic to $\mathbb{R}$. On the other hand, one can show that G_V is isomorphic to G.

Lemma 51. *Let G be a topological group with symmetric neighborhood U of e. If U is connected* (*respectively path connected*) *then G_U is connected* (*respectively path connected*).

Proof. Since U is connected and $U^k \subset G_U$ is the continuous image of $U \times \cdots \times U$, U^k is connected for all k. Since $e \in U^k$ for all k and $G = \bigcup U^k$, a standard result from topology shows that G_U is connected. (The proof of path connectedness is similar.) □

From the definition of G_U it is clear that G_U is generated by U. Hence we obtain:

Corollary 52. *Suppose U has any of these properties: connected, locally connected, arcwise connected, locally arcwise connected, locally compact. Then G_U has the same property.*

The group G_U has the following simple but important extension property.

Proposition 53. *Let G, H be topological groups and $U \subset G$, $V \subset H$ be symmetric neighborhoods of e. Then any (local group) homomorphism $\phi : U \to V$ extends uniquely to a homomorphism $\phi' : G_U \to H_V$. Furthermore, if ϕ has any of the conditions open, surjective, local isomorphism, or isomorphism, then ϕ' inherits the same property.*

Proof. Define, for any $x = [x_1 \cdots x_n] \in G_U$, $\phi'(x) = [\phi(x_1) \cdots \phi(x_n)]$. It is easy to verify that ϕ' is a well-defined homomorphism. Uniqueness follows from the fact that G_U is generated by U. The remaining properties follow from the definition of the Schreier group. □

Letting $V = H$ in the above proposition we obtain the following statement:

Corollary 54. *Let G, H be topological groups and $U \subset G$ be a symmetric neighborhood of e. Then any (local group) homomorphism $\phi : U \to H$ extends uniquely to a homomorphism $\phi' : G_U \to H$. If ϕ is open (respectively a local isomorphism) then ϕ' is open (respectively a local isomorphism) onto the open subgroup of H generated by $\phi'(U)$.*

Remark 55. The Schreier group G_U is completely characterized up to isomorphism by the above property in the following sense: If H is a topological group containing an isomorphic copy of U as a symmetric neighborhood of e, and H has the property that every local group homomorphism defined on U extends to H, then H is isomorphic to G_U.

Corollary 56. *If two homomorphisms $\phi, \phi' : G_U \to H$ agree on U then they are identically equal.*

Corollary 57. *If G, H, K are topological groups and U, V, W are symmetric neighborhoods of e in G, H, K, respectively, such that there is a local group isomorphism $\psi : U \to V \times W$, then ψ extends to an isomorphism $\psi' : G_U \to H_V \times K_W$.*

Proof. Note that $V \times W$ is naturally uniquely identified with a neighborhood of e in $H_V \times K_W$. Then $H_V \times K_W$ is generated by $V \times W$, since any $([a_1 \cdots a_n], [b_1 \cdots b_m]) \in H_V \times K_W$ is equal to $([a_1], e) \cdots ([a_n], e)(e, [b_1]) \cdots (e, [b_m])$, where each $a_i \in V$ and

$b_i \in W$. By Corollary 54 we have a unique extension $\psi' : G_U \to H_V \times K_W$ that is a local isomorphism onto $H_V \times H_W$. Likewise, the natural monomorphisms $i_V : V \to U$, and $i_W : W \to U$ extend to homomorphisms $i'_V : H_V \to G_U$ and $i'_W : K_W \to G_U$. Let $\xi : H_V \times K_W \to G_U$ be defined by $\xi((a,b)) := \iota'_V(a) \cdot i'_W(b)$. Then since $\xi \circ \psi'$ is the identity when restricted to U, it must be the identity by Corollary 56. Therefore ψ' is injective and hence an isomorphism. □

Corollary 58. *Let U, V be symmetric open neighborhoods of e in G. Then if $V \subseteq U$, the inclusion of V into U extends to a unique (possibly not surjective!) local isomorphism $\phi_{UV} : G_V \to G_U$. If c is a V-chain in G then c is also a U-chain, and we have*

$$\phi_{UV}\big([c]_V\big) = [c]_U. \tag{1}$$

Remark 59. In Example 48 the homomorphism $\phi_{WU} : G_U \to G_W$ obviously cannot be surjective.

Proposition 60. *Let G, H be topological groups, $U \subset G$, $V \subset H$ be symmetric neighborhoods of e, and $\phi : G \to H$ be a homomorphism such that $\phi(U) \subset V$. If $\phi' : G_U \to H_V$ denotes the homomorphism from Proposition* 53 *then the following diagram commutes*:

$$\begin{tikzcd} G_U \arrow[r, "\phi_{GU}"] \arrow[d, "\phi'"] & G \arrow[d, "\phi"] \\ H_V \arrow[r, "\phi_{HV}"] & H \end{tikzcd}$$

Moreover, if G_U is locally generated then ϕ' is the unique homomorphism such that this diagram commutes.

Proof. If $x \in G_U$, then write $x = [x_1 \cdots x_n]$, where $x_i \in U$. So

$$\begin{aligned} \phi_{HV}\big(\phi'(x)\big) &= \phi_{HV}\big(\phi'([x_1]) \cdots \phi'([x_n])\big) = \phi_{HV}\big([\phi(x_1)] \cdots [\phi(x_n)]\big) \\ &= \phi_{HV}\big([\phi(x_1)]\big) \cdots \phi_{HV}\big([\phi(x_n)]\big) = \phi(x_1) \cdots \phi(x_n), \end{aligned}$$

where the last equality follows from the fact that ϕ_{HV} can be considered as the identity on V and $\phi(U) \subset V$. But the last quantity is equal to

$$\phi\big(\phi_{GU}([x_1])\big) \cdots \phi\big(\phi_{GU}([x_n])\big) = \phi\big(\phi_{GU}([x_1]) \cdots \phi_{GU}([x_n])\big) = \phi\big(\phi_{GU}(x)\big).$$

Now let $\phi'' : G_U \to H_V$ be a homomorphism such that $\phi_{HV} \circ \phi'' = \phi \circ \phi_{GU}$. Let $W \subset U$ be a neighborhood of e in G_U such that $\phi'(W) \subset V$ and $\phi''(W) \subset V$. Then from the fact that ϕ_{GU} and ϕ_{HV} are local group isomorphisms when restricted to U and V, respectively, it follows from the commutativity of the diagram that ϕ' and ϕ'' coincide on W. Finally, if G_U is locally generated (hence generated by W) it follows that $\phi'' = \phi'$ on G_U. □

Proposition 61. *Let G be a topological group. Then G is locally defined if and only if for every neighborhood V of e in G there exists a symmetric neighborhood $U \subset V$ of e in G such that $\phi_{GU} : G_U \to G$ is an isomorphism.*

Proof. Suppose G is locally defined. Then given V there is a symmetric neighborhood $U \subset V$ of e in G such that any (local group) homomorphism defined on U extends uniquely to G. In particular, the inclusion of U into G_U extends uniquely to a homomorphism $\psi : G \to G_U$. But $\phi_{GU} \circ \psi : G \to G$ is a homomorphism whose restriction to U is the identity, and so, by the uniqueness of extensions, must be the identity. Likewise, Corollary 56 implies that $\psi \circ \phi_{GU}$ is the identity, and so ϕ_{GU} is an isomorphism.

To prove the converse, let V be given and choose a symmetric neighborhood $U \subset V$ of e in G such that $\phi_{GU} : G_U \to G$ is an isomorphism. Given any (local group) homomorphism $\phi : U \to H$, H a topological group, there exists, by Corollary 54, a unique extension $\phi' : G_U \to H$ of ϕ. But then $\phi' \circ \phi_{GU}^{-1}$ provides the desired extension of ϕ to G. The extension is unique, since if $\phi'' : G \to H$ were another extension of ϕ, then $\phi'' \circ \phi_{GU}$ would violate the uniqueness of ϕ'. □

According to Proposition 61, there are arbitrarily small symmetric neighborhoods U of e in G such that ϕ_{GU} is an isomorphism. Since G_U is generated by U, G is also generated by U. Thus G is generated by arbitrarily small, and hence all, neighborhoods of e. In other words, a locally defined group is locally generated. Combining this with Lemma 26 we have shown:

Corollary 62. *If G is a coverable topological group then G is locally generated.*

Proposition 63. *Let H be locally defined, G be a topological group, and $\psi : H \to G$ be a homomorphism. Then for any symmetric neighborhood U of e in G there is a unique homomorphism $\psi^U : H \to G_U$ such that $\psi = \phi_{GU} \circ \psi^U$.*

Proof. Using Proposition 61, let W be an open neighborhood of e in H such that $\psi(W) \subset U$ and $\phi_{HW} : H_W \to H$ is an isomorphism, and H_W is locally generated by Corollary 62. Then by Proposition 60 there is a unique homomorphism $\psi' : H_W \to G_U$ such that $\phi_{GU} \circ \psi' = \psi \circ \phi_{HW}$. Let $\psi^U := \psi' \circ \phi_{HW}^{-1}$. Then clearly $\psi = \phi_{GU} \circ \psi^U$. Suppose $\psi'' : H \to G_U$ is another homomorphism such that $\psi = \phi_{GU} \circ \psi''$. Again using Proposition 61, let $V \subset W$ be a symmetric neighborhood of e in H such that $\psi''(V) \subset U$ and $\phi_{HV} : H_V \to H$ is an isomorphism, and again H_V is locally generated. Note that ϕ_{WV} is an isomorphism and

$$\phi_{WV} = \phi_{HW}^{-1} \circ \phi_{HV}.$$

By Proposition 60 (identifying $(G_U)_U$ with G_U) there is a unique homomorphism $\psi''' : H_V \to G_U$ such that $\psi'' \circ \phi_{HV} = \psi'''$. Now

$$\begin{aligned} \phi_{GU} \circ \left(\psi''' \circ \phi_{WV}^{-1}\right) &= \phi_{GU} \circ \psi'' \circ \phi_{HV} \circ \phi_{WV}^{-1} \\ &= \phi_{GU} \circ \psi'' \circ \phi_{HW} = \psi \circ \phi_{HW}. \end{aligned}$$

By the uniqueness of ψ', $\psi' = \psi''' \circ \phi_{WV}^{-1}$. We have

$$\psi'' = \psi''' \circ \phi_{HV}^{-1} = \psi' \circ \phi_{WV} \circ \phi_{HV}^{-1} = \psi' \circ \phi_{HW}^{-1} = \psi^U. \qquad \square$$

Lemma 64. *Let G be a topological group. Then for any open symmetric neighborhoods $V \subset U$ of e, the natural homomorphism $\phi_{UV}: G_V \to G_U$ is surjective if and only if there exists an open neighborhood $W \subset V$ of e such that $\phi_{UW}: G_W \to G_U$ is surjective.*

Proof. Necessity is obvious. If $U \subset V \subset W$, then since the homomorphisms $\phi_{WV} \circ \phi_{VU}$ and ϕ_{WU} are both uniquely determined by their restrictions to U (cf. Corollary 56), we have the equation

$$\phi_{UW} = \phi_{UV} \circ \phi_{VW} \tag{2}$$

from which sufficiency follows. $\square$

We will need the following results in Section 6. First we show that, under fairly general circumstances, U-homotopies can be lifted:

Lemma 65. *Let G, H be topological groups, $\phi: G \to H$ be an epimorphism, and $U \subset G$, $V \subset H$ be neighborhoods of e and $U = \phi^{-1}(V)$. Suppose c is a U-chain to $x \in G$ and let $d := \phi(c)$. Then d is a V-chain. If d' is a V-chain to $y := \phi(x) \in H$, and h is a V-homotopy between d and d', then h lifts to a U-homotopy between c and some U-chain c'. That is, there exist a U-chain c' and a U-homotopy k between c and c' such that $\phi(k) = h$.*

Proof. Let $c := \{x_0, \ldots, x_n\}$. By definition, $x_i^{-1} x_{i+1} \in U$, so, letting $y_i := \phi(x_i)$, we have $y_i^{-1} y_{i+1} \in \phi(U) = V$ and $d = \phi(c)$ is a V-chain. For the remainder of the proof it suffices to consider the case when d' is V-related to d. Suppose first that d' is a V-extension of d; specifically, suppose that

$$d' = \{y_0, \ldots, y_i, z, y_{i+1}, \ldots, y_n\}.$$

Let $w \in \phi^{-1}(z)$. To complete the proof we need only show that

$$c' := \{x_0, \ldots, x_i, w, x_{i+1}, \ldots, x_n\}$$

is a U-chain. But $\phi(x_i^{-1} w) = y_i^{-1} z \in V$, since d' is a V-chain; then $x_i^{-1} w \in \phi^{-1}(V) = U$. Likewise $w^{-1} x_{i+1} \in U$ and the proof of this case is finished.

Now suppose d is a V-extension of d'; specifically, suppose that

$$d' = \{y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n\}.$$

We need to show that

$$c' := \{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$$

is a U-chain, i.e., that $x_{i-1}^{-1} x_{i+1} \in U$. But this follows as in the previous case. $\square$

Proposition 66. *Let G and H be topological groups, $\phi: G \to H$ be an epimorphism, $V \subset H$ be a symmetric neighborhood of e, and $U := \phi^{-1}(V)$. Let $\phi': G_U \to H_V$ denote* *the homomorphism given by Proposition* 53*. Finally, suppose $\phi_{GU} : G_U \to G$ is surjective and there exists a homomorphism $\psi : G \to H_V$ such that the following diagram commutes*:

$$
\begin{tikzcd}
G_U \arrow[r, "\phi_{GU}"] \arrow[d, "\phi'"'] & G \arrow[d, "\phi"] \arrow[dl, "\psi"] \\
H_V \arrow[r, "\phi_{HV}"] & H
\end{tikzcd}
\tag{3}
$$

Then ϕ_{GU} is an isomorphism.

Proof. Since ϕ_{GU} is assumed surjective, we need only show it is injective. Let $x \in \ker \phi_{GU}$. By Lemma 46, x corresponds to a U-chain $c := \{x_0, \ldots, x_n\}$ to e in G (i.e., a "U-loop"). Lemma 46 implies that we need only show that c is U-homotopic to the trivial chain $\{e\}$. Letting $y_i = \phi(x_i)$ and applying Lemma 65, $d = \{y_0, \ldots, y_n\}$ is a V-loop in H. We claim that d corresponds to $\phi'(x) \in H_V$. In fact, $x = [a_1 \cdots a_n]$ where $a_i = x_{i-1}^{-1} x_i \in U$, and since $\phi' = \phi$ on U, $\phi'(x) = [\phi(a_1) \cdots \phi(a_n)]$. But $y_{i-1}^{-1} y_i = \phi(x_{i-1}^{-1} x_i) = \phi(a_i)$ and the claim is proved. However, $\phi'(x) = \psi(\phi_{GU}(x)) = e$, so in fact d must be V-homotopic to the trivial chain. By the second part of Lemma 65, the V-homotopy between d and e lifts to a U-homotopy between c and chain c' lying in $\ker \phi \subset U$. Clearly the chain c' is U-homotopic to the trivial chain, and the proof is finished. □

Corollary 67. *Let H and G be topological groups with G locally generated, $\phi : G \to H$ be an epimorphism, V be a symmetric neighborhood of e in H, and $U := \phi^{-1}(V)$. If $\phi_{HV} : H_V \to H$ is an isomorphism then $\phi_{GU} : G_U \to G$ is an isomorphism.*

Proof. Define $\psi := \phi_{HV}^{-1} \circ \phi$. Then certainly $\phi_{HV} \circ \psi = \phi$. Let $\phi' : G_U \to H_V$ denote the homomorphism given by Proposition 53. Then by Proposition 60 we have $\phi_{HV} \circ \phi' = \phi \circ \phi_{GU}$. Now $\phi_{HV} \circ \phi' = \phi_{HV} \circ \psi \circ \phi_{GU}$, and applying ϕ_{HV}^{-1} to each side of the equation we get that the diagram (3) commutes. Since G is locally generated, ϕ_{GU} is surjective and the proof is complete by Proposition 66. □

Corollary 68. *Let $(G_\alpha, \pi_{\alpha\beta})$ be an inverse system of locally defined groups, where each $\pi_{\alpha\beta}$ is open. If the natural homomorphisms $\pi_\alpha : G \to G_\alpha$ are surjective then $G := \varprojlim G_\alpha$ is locally defined.*

Proof. Since each G_α is locally defined, it is locally generated (Corollary 62) and each $\pi_{\alpha\beta}$ is surjective. By Lemma 40, each π_α is an open surjection. Therefore by Lemma 41, G is locally generated. Consider a basis element of the topology of G at e of the form $U := \pi_\alpha^{-1}(V)$, where V is open in G_α such that $\phi_{G_\alpha V} : (G_\alpha)_V \to G_\alpha$ is an isomorphism. Now the conditions of Corollary 67 (replacing ϕ by ϕ_α) are satisfied and $\phi_{GU} : G_U \to G$ is an isomorphism. The proof is complete by Proposition 61. □

From Lemma 39 and the previous corollary we obtain:

Corollary 69. *Let* (G_i, π_{ij}) *be a* (*countable*) *inverse sequence of locally defined groups, where each* π_{ij} *is open. Then* $G := \varprojlim G_i$ *is locally defined.*

5. Properties of $\widetilde{G}$

Observe that Corollary 58, together with Eq. (2) of Lemma 64, provides an inverse system $\{G_U, \phi_{UV}\}$ indexed by the directed set $\mathcal{U}$ of symmetric open sets about e, partially ordered by reverse inclusion. Recall that the bonding homomorphisms ϕ_{UV} $(V \subset U)$ are (possibly not surjective!) local isomorphisms.

Definition 70. The collection $\{G_U, \phi_{VU}\}$ is called the inverse system of G. We denote by $\widetilde{G}$ the group $\varprojlim(G_U, \phi_{VU})$, and by $\phi_U : \widetilde{G} \to G_U$ the natural homomorphism.

Note that if G is complete then $\widetilde{G}$ is also complete (the product of complete groups, and a closed subgroup of a complete group are complete—see [6]). From Proposition 61 and Definition 70 we immediately obtain:

Corollary 71. *If* G *is locally defined, then the natural homomorphism* $\phi : \widetilde{G} \to G$ *is an isomorphism.*

Remark 72. In light of Corollary 58, we see that $\widetilde{G}$ consists of all elements $([c_U]_U)$ of $\prod_U G_U$ such that whenever $V \subset U$, c_U is a U-chain U-homotopic to c_V.

Theorem 73. *Let* H *and* G *be topological groups and* $\psi : H \to G$ *be a homomorphism. For any symmetric neighborhood* U *of* e *in* G, *let* $\psi_U : H_{\psi^{-1}(U)} \to G_U$ *be the unique homomorphism extending the restriction of* ψ *to* $\psi^{-1}(U)$. *Then there is a unique homomorphism* $\widetilde{\psi} : \widetilde{H} \to \widetilde{G}$ *such that for all* U, *if* $\phi_U : \widetilde{G} \to G_U$ *and* $\eta_W : \widetilde{H} \to H_W$ *denote the natural homomorphisms, then* $\psi_U \circ \eta_{\psi^{-1}(U)} = \phi_U \circ \widetilde{\psi}$. *In fact,*

$$\widetilde{\psi}\big(([c_V]_V)\big) = \big([\psi(c_{\psi^{-1}(U)})]_U\big). \tag{4}$$

If K *is another topological group and* $\zeta : G \to K$ *is a homomorphism then* $\widetilde{\zeta \circ \psi} = \widetilde{\zeta} \circ \widetilde{\psi}$. *If* ψ *is an isomorphism then* $\widetilde{\psi}$ *is an isomorphism.*

Proof. For any symmetric neighborhood U of e in G, let $U' := \psi^{-1}(U)$. Since ψ is a homomorphism, if c is a U'-chain in H then $\psi(c)$ is a U-chain in G. Likewise, if c and d are U'-homotopic U'-chains in H then $\psi(c)$ and $\psi(d)$ are U-homotopic U-chains in G. Therefore we have a well-defined homomorphism $[c]_{U'} \mapsto [\psi(c)]_U$ from $H_{U'}$ into G_U. The restriction of this homomorphism to U' coincides with the restriction of ψ to U', so by the uniqueness of ψ_U we must have $\psi_U([c]_{U'}) = [\psi(c)]_U$. Now suppose $V \subset U$, let $V' = \psi^{-1}(V)$, and let $[d]_{V'} \in H_{V'}$. Then from formula (1) in Corollary 58 we have

$$\psi_U \circ \eta_{U'V'}\big([d]_{V'}\big) = \psi_U\big([d]_{U'}\big) = \big[\psi(d)\big]_U = \phi_{UV}\big([\psi(d)]_V\big) = \phi_{UV} \circ \psi_V\big([d]_{V'}\big).$$

By the universal property of inverse limits there is a unique homomorphism $\widetilde{\psi}:\widetilde{H}\to\widetilde{G}$ such that for all U, $\psi_U\circ\zeta_{U'}=\phi_U\circ\widetilde{\psi}$, whose formula is given by (4).

To prove the second statement, note that

$$\begin{aligned}\widetilde{\zeta\circ\psi}\big(([c_V]_V)\big) &= \big([\zeta(\psi(c_{\psi^{-1}(\zeta^{-1}(W))}))]_W\big)=\widetilde{\zeta}\big(([\psi(c_{\psi^{-1}(U)})]_U)\big)\\ &= \widetilde{\zeta}\circ\widetilde{\psi}\big(([c_V]_V)\big),\end{aligned}$$

where U, V, W are symmetric neighborhoods of e in G, H, K, respectively.

If ψ is an isomorphism then from uniqueness it follows that $\widetilde{\psi}$ and $\widetilde{\psi^{-1}}$ are inverses of one another. □

Corollary 74. *If G is a topological group then $\phi(\widetilde{G})$ is a characteristic and fully characteristic subgroup of G.*

Notation 75. We will refer to the homomorphism $\widetilde{\psi}:\widetilde{H}\to\widetilde{G}$ in the above theorem as the homomorphism induced by ψ, or simply the induced homomorphism of ψ.

Theorem 76. *Let G be a topological group and $\phi:\widetilde{G}\to G$ be the natural homomorphism. Then for any locally defined group H and homomorphism $\psi:H\to G$ there exists a unique "lift" homomorphism $\psi':H\to\widetilde{G}$, such that $\psi=\phi\circ\psi'$.*

Proof. By Corollary 71, the natural homomorphism $\eta:\widetilde{H}\to H$ is an isomorphism. Let $\psi'=\widetilde{\psi}\circ\eta^{-1}$, where $\widetilde{\psi}:\widetilde{H}\to\widetilde{G}$ is the homomorphism induced by ψ. Then certainly $\psi=\phi\circ\psi'$. To prove uniqueness, suppose $\psi'':H\to\widetilde{G}$ is a homomorphism such that $\phi\circ\psi''=\psi$. According to Proposition 63, for any symmetric neighborhood U of e in G there is a unique homomorphism $\psi^U:H\to G_U$ such that

$$\phi_{GU}\circ\psi^U=\psi. \tag{5}$$

If If $\eta_{\psi^{-1}(U)}:\widetilde{H}\to H_{\psi^{-1}(U)}$ is the natural homomorphism and $\psi_U:H_{\psi^{-1}(U)}\to G_U$ is the homomorphism defined in Theorem 73 then it follows from Proposition 60 that $\psi_U\circ\eta_{\psi^{-1}(U)}\circ\eta^{-1}$ satisfies Eq. (5). Since $\phi_U\circ\psi''$ also satisfies Eq. (5), we must have $\psi_U\circ\eta_{\psi^{-1}(U)}\circ\eta^{-1}=\psi^U=\phi_U\circ\psi''$. It follows that $\psi_U\circ\eta_{\psi^{-1}(U)}=\phi_U\circ\psi''\circ\eta$, and so by the uniqueness of Theorem 73, $\psi''\circ\eta=\widetilde{\psi}$. Therefore $\psi''=\psi'$. □

Proposition 77. *Suppose G is a topological group such that $\widetilde{G}$ is locally generated. If $\phi:\widetilde{G}\to G$ is the natural homomorphism then* $\ker\phi$ *is central and prodiscrete.*

Proof. Let $V:=\phi_U^{-1}(W)$, W a neighborhood of e in G_U, U a symmetric neighborhood of e, be a basis element at the identity of the topology of $\widetilde{G}$. Note that $\ker\phi_U$ is a subgroup of $\ker\phi$, is contained in $V\cap\ker\phi$, and is normal in $\widetilde{G}$. We will apply Lemma 35; we first need to show that $\ker\phi_U$ is open in $\ker\phi$. Let ξ_U denote the restriction of ϕ_U to $\ker\phi$. Then $\xi_U(\ker\phi)\subset\ker\phi_{GU}$, which is discrete. That is, $\{e\}$ is open in $\ker\phi_{GU}$, and so $\ker\phi_U=\xi_U^{-1}(\{e\})$ is open in $\ker\phi$.

We now need to prove $\ker\phi$ is complete. By the definition of $\ker\phi$ and inverse limit,

$$\ker\phi = \left\{(a_U) \in \widetilde{G}\colon\ a_G = e\right\} = \left\{(a_U) \in \widetilde{G}\colon\ a_U \in \ker\phi_{GU}\right\} \subset \prod_U \ker\phi_{GU}.$$

The latter group is a product of discrete groups and so is complete. The closed subgroup $\ker\phi$ is then also complete. Now by Lemma 35, $\ker\phi$ is central and prodiscrete. □

If G is metrizable, then we can choose a countable, nested basis at e in G using finite intersections. According to Proposition 38 we can construct $\widetilde{G}$ using this basis. In other words, $\widetilde{G}$ is a subgroup of the countable product of metrizable groups, and we obtain:

Proposition 78. *If G is metrizable then $\widetilde{G}$ is metrizable.*

Proposition 79. *If G is a topological group then the arcwise connected component of G is contained in $\phi(\widetilde{G})$. In particular, if G is arcwise connected then $\phi:\widetilde{G} \to G$ is surjective. If $\widetilde{G}$ is arcwise connected then $\phi(\widetilde{G})$ is equal to the arcwise connected component of G.*

Proof. Since $\phi_{GU}:G_U \to G$ is an open homomorphism with discrete kernel, it is a (traditional!) covering epimorphism onto an open subgroup containing $(G)_e$. However, any curve $c:[0,1] \to G$ starting at e must remain in $(G)_e$ and therefore has a unique lift c_U to G_U starting at e such that $\phi_{GU} \circ c_U = c$. By uniqueness and the relation $\phi_{GV} = \phi_{GU} \circ \phi_{UV}$ it follows that for any $V \subset U$, $\phi_{UV}(c_V) = c_U$; we can apply the universal property of inverse limits to c to obtain a unique curve $\widetilde{c}:[0,1] \to \widetilde{G}$ such that $\widetilde{c}(0) = e$ and $\phi_U \circ \widetilde{c} = c_U$ for all U. In particular, $\phi \circ \widetilde{c} = c$. Now suppose $c':[0,1] \to \widetilde{G}$ is any curve such that $c'(0) = e$ and $\phi \circ c' = c$. Then $\phi_{GU} \circ \phi_U(c') = c$, so $\phi_U(c')$ is a lift of c. Therefore $\phi_U(c') = c_U$, and by uniqueness $c' = \widetilde{c}$. Therefore we have shown the existence of a unique curve $\widetilde{c}:[0,1] \to \widetilde{G}$ such that $\widetilde{c}(0) = e$ and $\phi \circ \widetilde{c} = c$. In particular, $c(1) \in \phi(\widetilde{G})$. □

Note that the main fact used in the above argument is that the bonding maps of the inverse system are traditional covers. In general, covers always give rise to such inverse systems:

Proposition 80. *Let $\psi:G \to H$ be a cover between topological groups. Then G is isomorphic to $\varprojlim(G/K_\alpha, \pi_{\alpha\beta})$, where $\{K_\alpha\}$ is the collection of open subgroups of* $\ker\psi$ *and $\pi_{\alpha\beta}$ is the natural epimorphism* (*which is open with discrete kernel*), *for $K_\beta \subset K_\alpha$.*

Proof. Note that since $\ker\psi$ is central, each K_α must be normal in G. Let $H_\alpha := G/K_\alpha$, $\pi_\alpha:G \to H_\alpha$ be the quotient epimorphism. Since $\ker\psi$ is prodiscrete $\{K_\alpha\}$ is a basis for the topology of $\ker\psi$ at e; it follows that $\bigcap K_\alpha = \{e\}$. Since $\ker\psi$ is complete, we have by [6, III.7.3 Proposition 2], that G is naturally isomorphic to $\varprojlim(H_\alpha, \pi_{\alpha\beta})$. Since each K_α is open in K and hence in K_β, for $\alpha \leqslant \beta$, $\ker\pi_{\alpha\beta} \equiv \ker K_\beta / \ker K_\alpha$ is discrete. □

Using the same method as in the proof of Proposition 79, together with Proposition 80 we can easily prove the following:

Proposition 81. *Let G, H be topological groups and $\psi : G \to H$ be a cover. Suppose X is a connected, locally arcwise connected, simply connected topological space. If $f : (X, p) \to (H, e)$ is a continuous function then there is a unique lift $g : (X, p) \to (G, e)$ such that $f = \psi \circ g$.*

Remark 82. Proposition 81 illustrates an important difference between covers and open epimorphisms with totally disconnected kernel. In fact, there is an open epimorphism with totally disconnected kernel $\psi : H \to T^\omega$, where H is separable Hilbert space, such that some curves in T^ω cannot be lifted to H (cf. [4]).

Although $\phi : \widetilde{G} \to G$ may not be a cover, we can still use the arguments from the proof of Proposition 79 to prove:

Proposition 83. *Let G be a topological group and X be a connected, locally arcwise connected, simply connected topological space. If $f : (X, p) \to (G, e)$ is a continuous function then there is a unique lift $\widetilde{f} : (X, p) \to (\widetilde{G}, e)$ such that $f = \phi \circ \widetilde{f}$.*

We now consider the relation between $\ker\phi$ and the traditional (Poincaré) fundamental group of G, which we refer to as $\pi_1^T(G)$. By definition, $\pi_1^T(G)$ consists of all homotopy equivalence classes of loops (based at e). For any loop $\gamma : [0,1] \to G$ based at e, by Proposition 83 there is a unique lift of γ to a curve $\widetilde{\gamma}$ starting at e in $\widetilde{G}$, and $\gamma(1) \in \ker\phi$. If γ' is a loop homotopic to γ, we can also lift a homotopy joining the two loops to $\widetilde{G}$, and it follows that $\gamma(1) = \gamma'(1)$. We therefore have a well-defined function $f : \pi_1^T(G) \to \ker\phi$. It is well known that in any topological group the concatenation of two loops is (up to reparameterization) homotopic to their product under the group operation, therefore f is a homomorphism.

Proposition 84. *Let G be a topological group. If $\widetilde{G}$ is arcwise connected then the natural homomorphism $f : \pi_1^T(G) \to \ker\phi$ is surjective. If f is surjective and G is arcwise connected, then $\widetilde{G}$ is arcwise connected. Finally, f is injective if and only if $\pi_1^T(\widetilde{G}) = e$.*

Proof. If $\widetilde{G}$ is arcwise connected then any $x \in \ker\phi$ can be joined to e via a curve γ. By uniqueness, the loop $\phi(\gamma)$, which represents an element α of $\pi_1^T(G)$, must lift to γ. Therefore $f(\alpha) = x$. Conversely, if f is surjective then every element of $\ker\phi$ can be joined to e in $\widetilde{G}$ via a lifted arc. Now suppose $x \in \widetilde{G}$ is arbitrary and G is arcwise connected. Join $\phi(x)$ to e by a curve and lift it to a curve in $\widetilde{G}$ from e to some $y \in \widetilde{G}$. But then $x^{-1}y \in \ker\phi$, and so x and y can be joined by the translate of a curve from e to $x^{-1}y$. We have therefore joined x and e by a curve.

Suppose that f is injective. Then every loop γ in $\widetilde{G}$ projects to a loop in G which is null-homotopic in G. But then this homotopy can be lifted, showing that γ is null-homotopic. The converse is trivial. □

Corollary 85. *If G is a topological group, and $\widetilde{G}$ is arcwise connected and $\pi_1^T(\widetilde{G}) = e$, then $\pi_1^T(G)$ is abstractly isomorphic to $\ker\phi$.*

Using Proposition 83 we can lift maps of higher dimensional spheres S^n, $n \geqslant 2$, and their homotopies. We obtain:

Corollary 86. *If G is a topological group then for all $n \geqslant 2$, ϕ induces an isomorphism from $\pi_n(\widetilde{G})$ onto $\pi_n(G)$.*

Proof of Theorem 12. This result is a consequence of a fact well-known to category theory experts, namely that certain functors commute with limits. Nonetheless, for sake of non-experts we provide a few more details. We will denote by $\{U_\gamma\}_{\gamma\in\Gamma}$ a basis for the topology of G at e consisting of symmetric open sets, with $U_{\gamma_0} = G$, and let $U_{\delta\gamma} := p_\delta(U_\gamma) \subset G_\delta$. Since each p_δ is open by Lemma 40, each collection $\{U_{\delta\gamma}\}_{\gamma=1,2,\ldots}$ is a basis for the topology at e in G_δ. Let $G_{\delta\gamma} := (G_\delta)_{U_{\delta\gamma}}$. We will construct a "double" inverse system as follows: For "horizontal" maps we have, for fixed δ and $\alpha \leqslant \gamma$, $h^\delta_{\alpha\gamma} := \phi_{U_{\delta\alpha}U_{\delta\gamma}} : G_{\delta\gamma} \to G_{\delta\alpha}$. Then by definition, for any δ, $\widetilde{G}_\delta = \varprojlim_\alpha (G_{\delta\alpha}, h^\delta_{\alpha\gamma})$; we denote by $h^\delta_\gamma : \widetilde{G}_\delta \to G_{\delta\gamma}$ the natural homomorphism. If β is fixed and $\alpha \leqslant \gamma$ then

$$p_{\alpha\gamma}(U_{\gamma\beta}) = p_{\alpha\gamma}\big(p_\gamma(U_\beta)\big) = p_\alpha(U_\beta) = U_{\alpha\beta}$$

so by Proposition 53, there is a unique open surjection $v^\beta_{\alpha\gamma} : G_{\gamma\beta} \to G_{\alpha\beta}$ extending the restriction of $p_{\alpha\gamma}$ to $U_{\gamma\beta}$. By Proposition 60 we have the following commutativity relation, for $\beta \leqslant \delta$:

$$h^\beta_{\alpha\gamma} \circ v^\gamma_{\beta\delta} = v^\alpha_{\beta\delta} \circ h^\delta_{\alpha\gamma}. \tag{6}$$

This commutativity relation determines a commutative "double" inverse system involving the groups $G_{\delta\gamma}$, parameterized by the set $\Delta \times \Gamma$, where Δ is the indexing set for the inverse system. Note that $\Delta \times \Gamma$ has a natural partial order with which it is a directed set. Let the inverse limit of this double inverse system be denoted by G''. By the universal property of the inverse limit, for every $\alpha \leqslant \gamma$ there is a unique homomorphism $\pi_{\alpha\gamma} : \widetilde{G}_\gamma \to \widetilde{G}_\alpha$ such that for any δ, $v^\delta_{\alpha\gamma} \circ h^\gamma_\delta = h^\alpha_\delta \circ \pi_{\alpha\gamma}$; it follows from the relation (6) that the homomorphisms $\pi_{\alpha\gamma}$ commute with all homomorphisms in the double diagram. Note that, by the uniqueness part of Theorem 73, $\pi_{\alpha\delta}$ must coincide with the homomorphism $\widetilde{p}_{\alpha\delta}$, and so G' is identified with $\varprojlim(\widetilde{G}_\alpha, \pi_{\alpha\delta})$.

For each γ we have a "vertical" inverse system with bonding homomorphisms $v^\gamma_{\delta\alpha}$; we let $G_{\infty\gamma} := \varprojlim_\delta (G_{\delta\gamma}, v^\gamma_{\delta\alpha})$ and denote by $v^\gamma_\delta : G_{\infty\gamma} \to G_{\delta\gamma}$ the natural homomorphism. As in the previous paragraph, there are unique homomorphisms $q_{\alpha\beta} : G_{\infty\beta} \to G_{\infty\alpha}$ that commute with all homomorphisms in the double diagram. It is not hard to prove the existence of natural isomorphisms $\tau_1 : G' \to G''$ and $\tau_2 : G'' \to \varprojlim G_{\infty\gamma}$. We therefore need to show the existence of a natural isomorphism $\tau_3 : \varprojlim G_{\infty\gamma} \to \widetilde{G}$. To do this, for fixed γ, consider the (local group) homomorphism $\mu'_\gamma : U_\gamma \to G_{\infty\gamma}$ given by

$$\mu'_\gamma\big((a_\alpha)\big) = \big([a_\alpha]\big).$$

Then μ'_γ is certainly a local group isomorphism onto its image, and so extends to a unique local isomorphism $\mu_\gamma : G_{U_\gamma} \to G_{\infty\gamma}$. Now the one-to-one homomorphism that sends the equivalence class $[((a_1)_\alpha) \cdots ((a_j)_\alpha)]$ to the element $(w_\alpha) \in G_{\infty\gamma}$, where w_α

is word-equivalence class $[(a_1)_\alpha \cdots (a_j)_\alpha]$, is an extension of μ'_γ. By uniqueness this homomorphism must be μ_γ, and so μ_γ must in fact be an isomorphism. For any $(a_\alpha) \in U_\delta$, by definition,

$$\mu_\gamma \circ \phi_{U_\gamma U_\delta}\big((a_\alpha)\big) = \mu_\gamma\big((a_\alpha)\big) = \big([a_\alpha]\big) = q_{\gamma\delta}\big(([a_\alpha])\big) = q_{\gamma\delta} \circ \mu_\delta\big((a_\alpha)\big).$$

Since we have the commutativity relation $\mu_\gamma \phi_{U_\gamma U_\delta} = q_{\gamma\delta} \circ \mu_\delta$ on U_δ, then by Corollary 56, $\mu_\gamma \phi_{U_\gamma U_\delta} = q_{\gamma\delta} \circ \mu_\delta$ on G_{U_δ}. From this commutativity relation we can produce from the isomorphisms μ_γ^{-1} the isomorphism τ_3.

To complete the proof, let $\eta = \tau_3 \circ \tau_2 \circ \tau_1$. Then $\eta : G' \to \widetilde{G}$ is an isomorphism. One can now verify that η is natural in the sense that $\phi \circ \eta = \phi'$, where $\phi' : G' \to G$ is defined by the sequence of natural homomorphisms $\phi_i : \widetilde{G}_i \to G_i$, and η is unique with respect to this property. □

Lemma 87. *Let G and H be topological groups, and $\psi : H \to G$ be an open epimorphism with discrete kernel. Then the homomorphism $\widetilde{\psi} : \widetilde{H} \to \widetilde{G}$ induced by ψ is an isomorphism.*

Proof. Let U be a symmetric neighborhood of e in H such that the restriction of ψ to U is a local group isomorphism. Note that the collection of all $\psi(V)$, where $V \subset U$ is a symmetric neighborhood of e in H, forms a basis for the topology of G at e. In other words, the collection of all such $\psi(V)$ is cofinal in the directed family of all symmetric neighborhoods of e in G, and we need only use such neighborhoods to determine $\widetilde{G}$. Then by definition $\widetilde{\psi}(([c_V]_V)) = ([\psi(c_{\psi^{-1}(\psi(V))})]_{\psi(V)})$. Since $V \subset \psi^{-1}(\psi(V))$, $c_{\psi^{-1}(\psi(V))}$ is $\psi^{-1}(\psi(V)$-homotopic to c_V, $\psi(c_{\psi^{-1}(\psi(V))})$ is $\psi(V)$-homotopic to $\psi(c_V)$, and we obtain $\widetilde{\psi}(([c_V]_V)) = ([\psi(c_V)]_{\psi(V)})$. From this formulation we see that $\widetilde{\psi}$ is defined by the homomorphisms $\zeta_V : H_V \to G_{\psi(V)}$ extending the restriction of ψ to V, which are all isomorphisms, and it follows that $\widetilde{\psi}$ is an isomorphism. □

Proposition 88. *If G and H are topological groups and $\psi : G \to H$ is a cover, then $\widetilde{G}$ is naturally isomorphic to $\widetilde{H}$.*

Proof. By Proposition 80, letting $H_\alpha := G/K_\alpha$, where K_α is an open subgroup of $\ker\psi$, we have that $G = \varprojlim(H_\alpha, \pi_{\alpha\beta})$, where $\pi_{\alpha\beta} : H_\beta \to H_\alpha$ is the natural epimorphism, for $K_\beta \subset K_\alpha$. By Theorem 41, $\widetilde{G}$ is naturally isomorphic to $\varprojlim(\widetilde{H}_\alpha, \widetilde{\pi}_{\alpha\beta})$. The natural open epimorphism $\psi_\alpha : H_\alpha \to H$ has discrete kernel (since K_α is open) and so $\widetilde{\psi}_\alpha : \widetilde{H}_\alpha \to \widetilde{H}$ is an isomorphism by Lemma 87. From $\psi_\beta = \psi_\alpha \circ \pi_{\alpha\beta}$ we obtain $\widetilde{\psi}_\beta = \widetilde{\psi}_\alpha \circ \widetilde{\pi}_{\alpha\beta}$, and so $\widetilde{\pi}_{\alpha\beta}$ is an isomorphism and each $\widetilde{H}_\alpha$ is isomorphic to $\widetilde{H}$. This completes the proof. □

6. Coverable groups

Definition 89. Let G be a topological group, U a symmetric neighborhood of e. Then U is called locally generated if G_U is locally generated.

Note that since every connected group is locally generated, by Corollary 52 we see that any connected symmetric neighborhood U of e is locally generated.

Theorem 90. *Let G be a topological group. The following are equivalent, where $\phi : \widetilde{G} \to G$ denotes the natural homomorphism*:

(1) *G is coverable.*
(2) *G has a basis for its topology at e consisting of locally generated symmetric neighborhoods, and ϕ is surjective.*
(3) *$\widetilde{G}$ is locally defined and ϕ is a cover.*

Proof. Suppose that G is coverable. By definition of coverable there exists a locally defined group H and an open epimorphism $\pi : H \to G$. By Theorem 76 there exists a unique homomorphism $\pi' : H \to \widetilde{G}$ such that $\phi \circ \pi' = \pi$. But since π is surjective, ϕ must be surjective. Now let V be a neighborhood of e in G. According to Proposition 61 there is a neighborhood W of e in H such that $\phi_{HW} : H_W \to H$ is an isomorphism and $U := \pi(W) \subset V$. Then the homomorphism $\pi'' : H_W \to G_U$ given by Proposition 53 is an open surjection. By definition, G_U is coverable, hence locally generated. We have shown (1) $\Rightarrow$ (2).

Suppose now that (2) holds. Then we can write $\widetilde{G} = \varprojlim G_U$ where each U is locally generated. By Lemma 42 (since $\phi : \widetilde{G} \to G = G_G$ is surjective), each of the homomorphisms ϕ_U is an open surjection and $\widetilde{G}$ is locally generated by Lemma 41. We will now prove that $\widetilde{G}$ is locally defined. Given any neighborhood U' of e in $\widetilde{G}$, we can find a basis element $U := \phi_V^{-1}(V) \subset U'$ for the topology of $\widetilde{G}$ at e, where V is a symmetric neighborhood of e in G. Now by the uniqueness part of Proposition 53, the natural homomorphism $\phi_{G_V V} : (G_V)_V \to G_V$ must be an isomorphism. Since we have already shown that $\widetilde{G}$ is locally generated, $\phi_{\widetilde{G}U} : \widetilde{G}_U \to \widetilde{G}$ is an isomorphism by Corollary 67, and the proof that $\widetilde{G}$ is locally defined is finished by Proposition 61.

If (2) holds then Lemma 42 implies that $\phi_U : \widetilde{G} \to G_U$ is an open surjection for any locally generated U. Since $\phi = \phi_{GU} \circ \phi_U$, ϕ is also open. Now (2) $\Rightarrow$ (3) follows from Proposition 77.

(3) $\Rightarrow$ (1) is immediate from the definition of coverable. $\square$

From Corollaries 52, 22, and Theorem 90 we have:

Corollary 91. *A locally connected group G is coverable if and only if $\phi : \widetilde{G} \to G$ is surjective. In this case G must be connected.*

For metrizable groups the situation is much simpler:

Theorem 92. *A metrizable group G is coverable if and only if G is locally generated and has a basis for its topology at e consisting of locally generated symmetric neighborhoods.*

Proof. To show sufficiency, choose a countable basis $\{U_i\}$ for the topology of G at e consisting of locally generated symmetric neighborhoods U_i such if $j \geqslant i$ then $U_j \subset U_i$, if and ϕ_{ij} denotes $\phi_{U_i U_j}$, then each ϕ_{ij} is an open surjection. It now follows from Lemma 39 that each $\phi_{U_i} : \widetilde{G} \to G_{U_i}$ is surjective, and sufficiency is proved by Theorem 90.

Necessity is an immediate consequence of Theorem 90 and the fact that a connected group is locally generated. □

Corollary 93. *Every metrizable, connected, locally connected group is coverable.*

Proof of Theorem 5. Since $\widetilde{G}_1$ is locally defined by Theorems 90 and 5 follows from applying Theorem 76 to the homomorphism $\psi \circ \phi_1$. □

Remark 94. We are justified in using the notation $\widetilde{\psi}$ introduced in Theorem 73 because the homomorphism given by Theorem 5 clearly satisfies the properties given in Theorem 73. By uniqueness, the two homomorphisms must coincide. In particular, formula (4) in Theorem 73 gives an explicit definition of $\widetilde{\psi}$.

Proposition 95. *If G, G' are locally generated, H is locally defined, $\psi : G \to G'$ is a local isomorphism and $\phi : H \to G'$ is a homomorphism, then there is a unique homomorphism $\eta : H \to G$ such that $\phi = \psi \circ \eta$.*

Proof. Since G' is locally generated, ψ is an epimorphism. Let U be a neighborhood of e in G such that ψ restricted to U is a (local group) isomorphism onto an open set V in G'. By Proposition 61 there is a neighborhood W of e in H such that $\phi(W) \subset V$ and $\phi_{HW} : H_W \to H$ is an isomorphism. Then we have a well-defined (local group) homomorphism $\eta' : W \to G$ given by $\eta'(x) = (\psi|_U)^{-1}(\phi(x))$. By Corollary 54, this homomorphism extends uniquely to a homomorphism $\eta'' : H_W \to G$; then $\psi \circ \eta'' = \phi \circ \phi_{HW}$. Let $\eta = \eta'' \circ \phi_{HW}^{-1}$. To prove uniqueness, let $\phi = \psi \circ \eta_1$, where $\eta_1 : H \to G'$ is a homomorphism. We need only show that $\eta_1 \circ \phi_{HW} = \eta'' : H_W \to G$. Since η'' and $\eta_1 \circ \phi_{HW}$ are uniquely determined by their restrictions to W, we need only verify $\eta_1 \circ \phi_{HW}(x) = \eta''(x)$ for any $x \in W$. But ϕ_{HW} restricted to W is the identity, and from $\phi = \psi \circ \eta_1$ we have $\eta_1 \circ \phi_{HW}(x) = \eta_1(x) = (\psi|_U)^{-1}(\phi(x)) = \eta'(x)$. □

If, in the above proposition, ϕ is open, then since $\phi = \psi \circ \eta$, η is open. If G is locally generated then ϕ is surjective. We have proved:

Corollary 96. *If G is locally generated, H is coverable, and $\psi : G \to H$ is a local isomorphism then G is coverable.*

Theorem 97. *Let H be a locally defined group, $\phi : G' \to G$ be a cover between locally generated topological groups, and $\psi : H \to G$ be a homomorphism. Then there exists a unique homomorphism $\psi' : H \to G'$ such that $\phi \circ \psi' = \psi$. Moreover, if ψ is open then the image of ψ' is dense in G'.*

Proof. Choose a family sequence $\{K_\alpha\}$ of open subgroups of the central subgroup $K :=$ $\ker \phi$ such that ϕ factors as $\phi = \phi_\alpha \circ \pi_\alpha$, where $\phi_\alpha : G_\alpha := G'/K_\alpha \to G$ is a surjective local isomorphism, and $\pi_\alpha : G' \to G_\alpha$ is the quotient epimomorphism. By Proposition 95, there is a unique homomorphism $\psi_\alpha : H \to G_\alpha$ such that $\phi_\alpha \circ \psi_\alpha = \psi$. If $\pi_{\alpha\beta} : G_\beta \to G_\alpha$

($\alpha \leqslant \beta$) denotes the natural homomorphism, then by uniqueness, for any $\beta \geqslant \alpha$, $\psi_\alpha = \pi_{\alpha\beta} \circ \psi_\beta$ and $\phi_\alpha \circ \pi_{\alpha\beta} = \phi_\beta$. Since K is complete, each K_α is complete, and so by [6, III.7.3, Proposition 2], G' is isomorphic to $\varprojlim G_\alpha$. By the universal property of the inverse limit (see Section 3) there exists a unique homomorphism $\psi' : H \to G'$ such that for all α, $\pi_\alpha \circ \psi' = \psi_\alpha$. To prove uniqueness, note that if $\psi'' : H \to G'$ is a homomorphism satisfying $\phi \circ \psi'' = \psi = \phi \circ \psi'$, then for all α,

$$\phi_\alpha \circ \pi_\alpha \circ \psi'' = \phi \circ \psi'' = \phi \circ \psi' = \phi_\alpha \circ \pi_\alpha \circ \psi' = \phi_\alpha \circ \psi_\alpha$$

so by the uniqueness of ψ_α, $\pi_\alpha \circ \psi'' = \psi_\alpha$. By the uniqueness part of the universal property of inverse limits, $\psi'' = \psi'$.

Now suppose that ψ is open, and therefore surjective (since G is locally generated). Let $x \in G'$ and U be an open neighborhood of x in G'. Since ϕ is a cover we can find a closed normal subgroup N of $\ker\phi$ contained in $x^{-1}U$ such that $\ker\phi/N$ is discrete. Let $G'' = G'/N$. Then we have $\phi = \eta \circ \pi_1$, where $\pi_1 : G' \to G''$ is the quotient epimomorphism and $\eta : G'' \to G$ is a surjective local isomorphism. We claim that $\pi_1 \circ \psi'$ is open. Let V be a neighborhood of e in G'' such that η restricted to V is homeomorphic onto its image $W := \eta(V)$. Let $V' \subset H$ be an open neighborhood of e such that $\pi_1(\psi'(V')) \subset V$. Then $\eta(V) \supset \eta(\pi_1 \circ \psi'(V')) = \psi(V')$, which is open since ψ is open by assumption. Since η restricted to V is a homeomorphism, $\pi_1 \circ \psi'(V')$ is open and it follows that $\pi_1 \circ \psi'$ is open and therefore surjective onto the locally generated group $G'' = G'/N$. In other words, there exists some $z \in H$ such that $\pi_1(\psi'(z)) = \pi_1(x)$. That is, if $y := \psi'(z)$, $x^{-1}y \in N \subset x^{-1}U$, so $y = \psi'(z) \in U$. This completes the proof of the theorem. $\square$

Remark 98. In the above proof we use for the first time the completeness of the kernel of a covering epimorphism, and it will not be explicitly used again (although many of the remaining results depend on Theorem 97).

Example 99. Let Σ be the 2-adic solenoid, i.e., the inverse limit of circles, with open bonding epimorphisms that are double coverings. It is well known that Σ is connected but not locally connected. From Theorem 12 it follows that $\widetilde{\Sigma}$ is the real numbers $\mathbb{R}$, and $\phi : \mathbb{R} \to \Sigma$ cannot be a cover. Thus Σ is not coverable by Theorem 90. Note that the circle is coverable, and so Σ shows that we cannot replace "locally defined" by "coverable" in Corollaries 68 and 69. We can recover ϕ as follows: Let $\xi : \Sigma \to S^1$ be the projection onto any factor. It is not hard to verify that ξ is a cover. Let $\psi : \mathbb{R} \to S^1$ be the universal covering homomorphism. According to Theorem 97 there exists a unique homomorphism $\eta : \mathbb{R} \to \Sigma$ such that $\xi \circ \eta = \psi$, and by uniqueness this homomorphism must coincide with ϕ. (Theorem 97 also correctly predicts that this homomorphism has dense image.) This shows that, in general, the homomorphism ψ' given by Theorem 97 may not be surjective, even if ψ is a surjective local isomorphism. This example also shows that in Corollary 96 one cannot replace "surjective local isomorphism" by "cover". Note that $\phi(\mathbb{R})$, with the subspace topology, is not coverable by Theorem 15. So the image of a coverable group by a (continuous) epimorphism need not be coverable. Since Σ is a closed subgroup of the connected, locally arcwise connected, compact (and therefore coverable) group $T^\omega = (S^1)^\omega$, this example also shows that a closed, locally generated subgroup of a coverable group need not be coverable.

Example 100. Consider the countably infinite product of reals, $\mathbb{R}^\omega$ and the closed, prodiscrete subgroup $\mathbb{Z}^\omega$. Let G denote the (not closed!) subgroup of $\mathbb{R}^\omega$ consisting of all sequences such that all but finitely many coordinates are zero, and let $K := G \cap \mathbb{Z}^\omega$. Then K has a countable basis $\{K_i\}$ for its topology at e consisting of open (normal) subgroups K_i. (For example, we can take K_i to be the subgroup of K consisting of all elements of K whose first i coordinates are 0.) However, neither G nor K is complete. Let $G_i := G/K_i$ and $\overline{G}$ denote $\varprojlim G_i$. Then according to [6, III.7.3, Proposition 2], there is a natural homomorphism $\iota : G \to \overline{G}$ that is an isomorphism onto a dense subgroup of $\overline{G}$. As in the proof of Proposition 77, the inverse limit $\overline{K}$ of the kernels of the natural homomorphisms $\phi_{ij} : G_j \to G_i$ is a prodiscrete subgroup of $\overline{G}$, and again by [6, III.7.3, Proposition 2], the restriction of ι to K is an isomorphism onto a dense subgroup of $\overline{K}$; in other words $\overline{K}$ is the completion of K. (One might refer to $\overline{G}$ as the K-completion of G.) We have two quotient epimorphisms

$$\phi_1 : G \to G/K \quad \text{and} \quad \phi_2 : \overline{G} \to \overline{G}/\overline{K} = G/K,$$

each having a kernel with a basis for its topology at e consisting of open normal subgroups. Now G is contractible, hence locally defined by Corollary 118. Since G is dense in $\overline{G}$, $\overline{G}$ is also locally defined by Corollary 129. By Corollary 11, $\overline{G}$ is naturally isomorphic to $\widetilde{G/K}$. But the homomorphism ι, which satisfies $\phi_1 = \phi_2 \circ \iota$ is not an isomorphism. This example shows that we cannot relax the requirement that the kernels of covers be complete (cf. Lemma 32) because $\phi_2 : \overline{G} \to G/K$ fails to have the universal property with respect to the open epimorphism $\phi_1 : G \to G/K$. In [14], Kawada defined a "generalized universal covering" to be an open epimorphism $\psi : A \to B$ between connected, locally connected groups such that A is simply connected in the sense of Chevalley [8], and $\ker \psi$ is central, totally disconnected, and has a basis for its topology at e consisting of open subgroups. Note that the last condition itself implies total disconnectedness, and, as we have recalled earlier, the centrality of $\ker \psi$ is already implied by the connectedness of A. Now the metrizable topological groups G and $\overline{G}$ constructed above are both locally defined and it follows (e.g., from Theorem 7) that every traditional cover of G or $\overline{G}$ is trivial. By definition, both groups are simply connected in the sense of Chevalley, and so both the homomorphisms $\phi_1 : G \to G/K$ and $\phi_2 : \overline{G} \to G/K$ are generalized universal covers in the sense of Kawada. The topological vector space G is connected and locally connected, so $\overline{G}$ is also connected and locally connected. This contradicts Kawada's uniqueness theorem [14, Theorem 4].

Theorem 101. *Suppose G, H are coverable groups, $\phi : \widetilde{H} \to H$ and $\phi' : \widetilde{G} \to G$ are the natural epimorphisms and $\pi : G \to H$ is a cover. Then there is a unique isomorphism $\eta : \widetilde{G} \to \widetilde{H}$ such that $\phi \circ \eta = \pi \circ \phi'$. Moreover, if $\psi := \phi' \circ \eta^{-1} : \widetilde{H} \to G$ then ψ is a cover, and is the unique cover such that $\phi = \pi \circ \psi$.*

Proof. We will first construct the following commutative diagram:

$$\begin{tikzcd} \widetilde{G} \arrow[r, shift left, "\eta"] \arrow[d, "\phi'"'] & \widetilde{H} \arrow[l, shift left, "\xi"] \arrow[d, "\phi"] \arrow[dl, "\psi"] \\ G \arrow[r, "\pi"] & H \end{tikzcd}$$

By Theorem 90, $\widetilde{G}$ is locally defined, so by Theorem 5 there exists a unique homomorphism $\eta : \widetilde{G} \to \widetilde{H}$ such that $\phi \circ \eta = \pi \circ \phi'$. Also, $\widetilde{H}$ is locally defined, so by Theorem 97 there exists a unique homomorphism $\psi : \widetilde{H} \to G$ such that $\pi \circ \psi = \phi$. Likewise there exists a unique homomorphism $\xi : \widetilde{H} \to \widetilde{G}$ such that $\phi' \circ \xi = \psi$. Note that $\pi \circ \psi \circ \eta = \phi \circ \eta = \pi \circ \phi'$. According to Theorem 97 there must be a unique homomorphism $\omega : \widetilde{G} \to G$ such that $\pi \circ \omega = \pi \circ \phi'$. Since both ϕ' and $\psi \circ \eta$ satisfy this property, $\phi' = \psi \circ \eta$ and the entire diagram is commutative. Since $\phi' \circ \xi \circ \eta = \psi \circ \eta = \phi'$, we have by the uniqueness of Theorem 97 that $\xi \circ \eta$ is the identity on $\widetilde{G}$. Likewise, $\phi \circ \eta \circ \xi = \pi \circ \phi' \circ \xi = \pi \circ \psi = \phi$, so by uniqueness $\eta \circ \xi$ is the identity on $\widetilde{H}$, and so η and ξ are inverses, and therefore isomorphisms. Now $\psi = \phi' \circ \xi$ is evidently an open epimorphism. By Theorem 90, $\ker \phi'$ is central and prodiscrete. Since $\ker \psi = \xi^{-1}(\ker \phi')$ and ξ is an isomorphism, $\ker \psi$ is also central and prodiscrete. Therefore ψ is the desired cover, whose uniqueness we have already proved. □

Corollary 102. *Suppose G, H are coverable groups, $\phi' : \widetilde{G} \to G$ is the natural epimorphism and $\pi : G \to H$ is a cover. If $\nu : \widetilde{G} \to \widetilde{G}$ is an isomorphism such that $\pi \circ \phi' \circ \nu = \pi \circ \phi'$ then ν must be the identity.*

Proof. Applying Theorem 101, and using its notation, $\phi \circ \eta \circ \nu = \pi \circ \phi' \circ \nu = \pi \circ \phi' = \phi \circ \eta$. By the uniqueness of η, $\eta = \eta \circ \nu$, and the proof is complete since η is an isomorphism. □

We do not know of a reference for the following simple result from general topology, which is useful for us.

Lemma 103. *Let $f : X \to Y$ be an open, onto function between topological spaces. Then for any $A \subset Y$, the restriction of f to $Z := f^{-1}(A)$ is an open onto function from Z onto A.*

Proof. Let V be open in Z; that is, $V = U \cap Z$ where U is open in X. Since f is open, it suffices to prove $f(V) = f(U) \cap A$. From set theory we know that $f(V) \subset f(U) \cap A$. Suppose that $y \in f(U) \cap A$. Then there exists an $x \in U$ such that $f(x) = y$. Since $y \in A$, it follows that $x \in f^{-1}(A) = Z$ and so $x \in Z \cap U = V$, and $y \in f(V)$. □

Proof of Theorem 6. First note that ψ and π are open epimorphisms, and therefore so is $\pi \circ \psi$. We need only show that $H := \ker(\pi \circ \psi)$ is prodiscrete and central. Consider the following commutative diagram of covers, where ϕ_2 and ϕ_3 are the natural epimorphism,

π_2 is the isomorphism provided by Theorem 101, and finally, π_4 is the cover provided by Theorem 101:

$$\begin{tikzcd}
 & G_1 \arrow[d, "\psi"] \\
\widetilde{G}_2 \arrow[ur, "\pi_4"] \arrow[r, "\phi_2"] \arrow[d, "\pi_2"] & G_2 \arrow[d, "\pi"] \\
\widetilde{G}_3 \arrow[r, "\phi_3"] & G_3
\end{tikzcd}$$

We have that $K := \ker(\phi_3 \circ \pi_2) = \pi_2^{-1}(\ker \phi_3)$ is prodiscrete and central. Now by the commutativity of the diagram, $\pi_4^{-1}(H) = K$. Since π_4 an open surjection, Lemma 103 implies that the restriction of π_4 to K is an open surjection onto H, which is therefore prodiscrete and central by Lemmas 33 and 35. $\square$

Proof of Theorem 7. By Theorem 101 there exists a unique isomorphism $\eta : \widetilde{G} \to \widetilde{H}$ such that $\phi \circ \eta = \pi \circ \phi'$, where $\phi' : \widetilde{G} \to G$ is the natural epimorphism. The desired cover $\psi : \widetilde{H} \to G$ is defined by $\psi := \phi' \circ \eta^{-1}$. Suppose $\psi' : \widetilde{H} \to G$ is another cover such that $\phi = \pi \circ \psi'$. Since $\widetilde{H}$ is locally defined, it follows from Corollary 71 that $\widetilde{\widetilde{H}}$ is isomorphic to $\widetilde{H}$, and so there is a unique isomorphism $\xi : \widetilde{H} \to \widetilde{G}$ such that $\phi' \circ \xi = \psi'$. Note that $\pi \circ \phi' \circ \xi \circ \eta = \pi \circ \psi' \circ \eta = \phi \circ \eta = \pi \circ \phi'$. By Corollary 102, $\xi = \eta^{-1}$ and $\psi' = \psi$. $\square$

Remark 104. In [4] we proved a universal property (and hence uniqueness) of simply connected (in the traditional sense) covers of complete connected, locally arcwise connected groups, but we proved existence of simply connected covers only in the metrizable locally compact case.

Proof of Theorem 8. First note that the induced map (from Theorem 5) $\widetilde{\pi} : \widetilde{G}_3 \to \widetilde{G}_2$ is an isomorphism, by Theorem 101. By Theorem 5 we have the following commutative diagram, where ϕ_i denotes the universal covering epimorphism:

$$\begin{tikzcd}
\widetilde{G}_1 \arrow[r, "\widetilde{\psi}"] \arrow[d, "\phi_1"] & \widetilde{G}_2 \arrow[r, "\widetilde{\pi}^{-1}"] \arrow[d, "\phi_2"] & \widetilde{G}_3 \arrow[d, "\phi_3"] \\
G_1 \arrow[r, "\psi"] & G_2 & G_3 \arrow[l, "\pi"']
\end{tikzcd}$$

The condition $\psi_*(\pi_1(G_1)) \subset \pi_*(\pi_1(G_3))$ implies that $\widetilde{\pi}^{-1}(\widetilde{\psi}(\ker \phi_1)) \subset \ker \phi_3$. If we define

$$\psi' : G_1 \to G_3 \quad \text{by } \psi'(x) = \phi_3\bigl(\widetilde{\pi}^{-1}(\widetilde{\psi}(y))\bigr),$$

where $x = \phi_1(y)$, then a standard diagram chase implies that ψ' is a well-defined homomorphism having the desired commutativity property. To see why ψ' is continuous, let $V \subset G_3$ be open. Then by definition of ψ',

$$(\psi')^{-1}(V) = \phi_1\bigl(\widetilde{\psi}^{-1}\bigl(\widetilde{\pi}(\phi_3^{-1}(V))\bigr)\bigr) = \psi^{-1}\bigl(\pi(V)\bigr),$$

which is open since ψ is continuous and π is open. Now suppose that ψ'' is another such lift. Then by Theorem 97 there exists a unique homomorphism $\psi''': \widetilde{G}_1 \to G_3$ such that $\pi \circ \psi''' = \psi \circ \phi_1$. Since both $\psi'' \circ \phi_1$ and $\phi_3 \circ \widetilde{\pi}^{-1} \circ \widetilde{\psi}$ have this property, $\phi_3 \circ \widetilde{\pi}^{-1} \circ \widetilde{\psi} = \psi''' = \psi'' \circ \phi_1$, and therefore $\psi' = \psi''$ by the definition of ψ'.

Conversely, suppose such a lift ψ' exists. By the functorial property of the induced homomorphism (which can easily be proved),

$$\psi_*\big(\pi_1(G_1)\big) = \pi_*\big(\psi'_*(\pi_1(G_1))\big) \subset \pi_*\big(\pi_1(G_3)\big).$$

Now suppose that ψ is a cover. Then the previous construction of ψ' means that for the isomorphism $i = \widetilde{\pi}^{-1} \circ \widetilde{\psi}$ and $K_j := \ker \phi_j$ $(j = 1, 2, 3)$, ψ' factors as the composition of two homomorphisms $\bar{i}: G_1 = \widetilde{G}/K_1 \to \widetilde{G}_3/i(K_1)$ and $p: \widetilde{G}_3/i(K_1) \to (\widetilde{G}/i(K_1))/(K_3/i(K_1))$, where the latter group is isomorphic to $\widetilde{G}_3/K_3 = G_3$. Here $\bar{i}$ is the natural isomorphism induced by the isomorphism i. The subgroup $i(K_1) \subset K_3$ must be central in $\widetilde{G}_3$ (Theorem 4) hence normal in K_3. Also, $i(K_1)$ is prodiscrete, hence complete (Lemma 32), hence closed in K_3. Then $K_3/i(K_1)$ is prodiscrete (Lemma 33) and the natural projection p is a cover. Hence $\psi' = p \circ \bar{i}$ is a cover. The last statement of the theorem follows from the previous one. □

Proof of Theorem 9. According to Theorem 101 there exists a unique isomorphism $\widetilde{\pi}: \widetilde{G} \to \widetilde{H}$ such that if $\phi: \widetilde{G} \to G$ and $\phi': \widetilde{H} \to H$ denote the universal covering epimorphisms, then $\pi \circ \phi = \phi' \circ \widetilde{\pi}$. Since $\widetilde{\pi}$ is an isomorphism, $\pi_*(\pi_1(G)) = \widetilde{\pi}(\ker \phi)$ is a closed subgroup of $\widetilde{H}$. Since $\pi \circ \phi = \phi' \circ \widetilde{\pi}$, $\pi_*(\pi_1(G))$ is a subgroup of (hence closed in) $\ker \phi' = \pi_1(H)$. Now suppose G is an arbitrary coverable group and K is a closed subgroup of the central subgroup $\pi_1(G) = \ker \phi$ of $\widetilde{G}$. Then K is central in $\widetilde{G}$ and prodiscrete by Lemma 33. Therefore the natural projection $\widetilde{G} \to \widetilde{G}/K := G'$ is a cover and G' is coverable. The natural projection $\pi: \widetilde{G}/K \to (\widetilde{G}/K)/(\pi_1(G)/K)) = \widetilde{G}/\pi_1(G) = G$ is also a cover, since $\pi_1(G)/K$ is prodiscrete and central by Lemma 33. Evidently $\widetilde{\pi}: \widetilde{G} \to \widetilde{G}$ is the identity, so $\pi_*(\pi_1(G')) = \widetilde{\pi}(K) = K$, as required. The uniqueness of $\pi: G' \to G$ up to isomorphism of covers follows from Theorem 8. □

To end this section we make a couple of general observations:

Proposition 105. *If G is coverable, H is any topological group, $\psi: G \to H$ is a homomorphism and $\phi: \widetilde{H} \to H$ is the natural homomorphism, then $\psi(G) \subset \phi(\widetilde{H})$.*

Proof. Let $\eta: \widetilde{G} \to G$ be the universal covering epimorphism. Since $\widetilde{G}$ is locally defined, Theorem 76 provides a unique homomorphism $\xi: \widetilde{G} \to \widetilde{H}$ such that $\phi \circ \xi = \psi \circ \eta$, and the proposition follows. □

Corollary 106. *If G is a topological group and $x \in G$ lies on a one-parameter subgroup then $x \in \phi(\widetilde{G})$, where $\phi: \widetilde{G} \to G$ is the natural homomorphism.*

Corollary 107. *If G is a locally connected topological group generated by its one-parameter subgroups then G is coverable.*

Proof. If G is generated by its one-parameter subgroups then by the previous corollary $\phi : \widetilde{G} \to G$ is surjective. Since G is locally connected we can now apply Theorem 90(2). □

7. Traditional covers

Throughout this section "cover" has its traditional meaning, namely an open epimorphism with discrete kernel. A "universal cover" for topological groups is a covering epimorphism $\phi : G \to H$ such that for any cover $p : F \to H$ of topological groups there is a homomorphism $\psi : G \to F$ such that $\phi = p \circ \psi$. If G and F are connected it follows easily that ψ is a cover and is unique. In this section we produce a new theory in the category of topological groups and covers, which includes as special cases the classical Poincaré covering group theory for connected, locally arcwise connected, semilocally simply connected groups (P), as well as covering group theories descended from that of Chevalley [8] (C) for groups that are connected locally connected, and locally simply connected in a sense we will explain below. In particular, generalizations of some of the results of Kawada [14] and Tits [24] are contained in the results below. Also included are the main results of [12, Appendix 2], in which Chevalley's theory is developed, in part without local connectedness. The special theory in this section fits nicely into our general framework, because in this category the universal cover turns out to be the homomorphism $\phi : \widetilde{G} \to G$ (Theorem 123).

We consider pointed topological spaces (X, p) and (pointed, continuous) maps $f : (X, p) \to (Y, q)$ (i.e., $f(p) = q$).

Definition 108. A map $i : (S, s) \to (T, t)$ between topological spaces is called simply connected if for every map $h : (T, t) \to (A, a)$ and every cover $p : (B, b) \to (A, a)$ there is a continuous function $\sigma : (S, s) \to (B, b)$ such that the following diagram commutes:

$$\begin{tikzcd} (S,s) \arrow[r, "\sigma"] \arrow[d, "i"] & (B,b) \arrow[d, "p"] \\ (T,t) \arrow[r, "h"] & (A,a) \end{tikzcd}$$

We say that S is simply connected if the identity map on S is simply connected. We say that S is semilocally simply connected if each point $s \in S$ is contained in a connected open set U such that the inclusion of (U, s) into (S, s) is simply connected.

Remark 109. By standard topological arguments it follows that if S is connected then σ is the unique such map; we will omit the details of these kinds of arguments in this section.

If S is connected, our definition that S is simply connected is identical to the one in [12]. If S is connected and locally arcwise connected, one can show using standard curve lifting arguments that if S is simply connected (respectively semilocally simply connected) in the traditional sense, then S satisfies the respective definition above. In [12], for any connected topological space X that is locally simply connected (i.e., X has a simply connected neighborhood about each point), a cover $\widetilde{p}:\widetilde{X} \to X$ is constructed, where $\widetilde{X}$ is simply connected. For a topological group G, $\widetilde{G}$ can be made, in the usual way, into a topological group such that $\widetilde{p}$ is a homomorphism. The resulting epimorphism $\widetilde{p}:\widetilde{G} \to G$ is a universal cover.

In keeping with the general approach of this paper, when considering topological groups we immediately include the algebraic structure in the definition, weakening the requirements:

Definition 110. If (T,t) is a topological group we always take $t = e$ and consider the requirements in Definition 108 only for topological groups (A,a) and $(B, b = e)$, homomorphism h, and covering epimorphism p.

Lemma 111. *Let G be a topological group, $p:H \to K$ be a cover of topological groups and $\phi:G \to K$ be a homomorphism. For any symmetric neighborhoods U, V of e in G such that V is connected and $V^2 \subset U$, if $f:U \to H$ is a continuous function such that $f(e) = e$ and $p \circ f = \phi$, then the restriction of f to V is a* (*local group*) *homomorphism, and is the unique such homomorphism on V.*

Proof. Consider the continuous function $\zeta:V \times V \to H$ given by $\zeta((x,y)) = f(xy)f(y)^{-1}f(x)^{-1}$ ($f(xy)$ is always defined because $xy \in V^2 \subset U$). Since ϕ and p are homomorphisms and $p \circ f = \phi$, $\zeta(V \times V)$ is contained in the discrete group $\ker p$. Since $V \times V$ is connected, $\zeta(V \times V) = e$ and the proof that f is a homomorphism is complete. Uniqueness follows from the fact that V is connected . □

Example 48 can be used to show that connectedness of V is cannot be simply removed in the above statement.

Lemma 112. *Let G and H be topological groups and $\phi:G \to H$ be a cover.*

(1) *If G is simply connected,then ϕ is simply connected.*
(2) *If G is connected and ϕ is simply connected then ϕ is a universal cover.*

Proof. The first statement follows simply from the definition. To show the second, let $p:K \to G$ be a cover, where K is a topological group. Since ϕ is simply connected there is a continuous function $f:G \to K$ such that $p \circ f$ is the identity. Letting $U := V := G$ in the previous lemma shows f is a homomorphism and completes the proof. □

The next theorem shows, in much generality, that connected universal covers can be constructed using only a Schreier group.

Theorem 113. *Let G be a connected topological group and U be a neighborhood of e in G such that the inclusion of U into G is simply connected. Then for any symmetric connected neighborhood V of e in G such that $V^2 \subset U$, $\phi_{GV}:G_V \to G$ is a simply connected cover, hence a universal cover* (*since G_V is connected*).

Proof. Let $p:K \to H$ be a cover and $\phi:G \to H$ be a homomorphism. Then by definition of simply connected there exists a continuous function $f:U \to K$ such that $p \circ f$ is the restriction to U of ϕ. By Lemma 111, f restricted to V is a homomorphism. By Corollary 54 the restriction of f to V extends to a homomorphism $\widetilde{f}:G_V \to K$ such that $p \circ \widetilde{f} = \phi \circ \phi_{GV}$. G_V is connected by Corollary 52. Since G is connected, ϕ_{GV} is a surjective, hence a cover, and the last statement follows from Lemma 112. □

Note that Example 49 shows that the assumption $V^2 \subset U$ is essential.

For any group G we denote by G_e the connected component of G containing e.

Lemma 114. *Let $h:\overline{G} \to G$ and $f:H \to \overline{G}$ be covers between topological groups. If $\overline{G}$ is connected and h is a universal cover then the restriction f' of f to H_e is an isomorphism onto $\overline{G}$, and H_e is open in H.*

Proof. Evidently $h \circ f$ is a cover. By definition of universal cover there is a continuous function $\phi:\overline{G} \to H$ such that $h \circ f \circ \phi = h$. By Lemma 111, ϕ is unique and is a homomorphism. By uniqueness again, $f \circ \phi$ is the identity on $\overline{G}$. Since f is a cover, ϕ is a local isomorphism, hence open. It now follows that $\phi(\overline{G}) = H_e$, which is therefore open in H, and $\phi:\overline{G} \to H_e$ and $f':H_e \to \overline{G}$ are inverse isomorphisms of one another. □

Lemma 115. *Let G be a topological group, $h:G \to H$ be a homomorphism and $p:K \to H$ be a cover. Then for any sufficiently small symmetric neighborhood W of e in G there is a unique homomorphism $\psi:G_W \to K$ such that $h \circ \phi_{GW} = p \circ \psi$.*

Proof. There exist neighborhoods U, V of e in K, H, respectively such that p restricted to U is a local group isomorphism onto V. Now simply choose W small enough that $h(W) \subset V$ and apply Corollary 54 to $(p|_V)^{-1} \circ h|_W$. □

We now consider Theorem II.VII.3 in [8], which we will show in the more general form of Corollary A2.26 of [12]. (Note that the errata list for [12] corrects the statements of Corollaries A2.26 and A2.28 by adding the assumption that U be connected—Example 48 in the present paper shows that connectedness cannot simply be removed. Note also that by modifying the extension results for Schreier groups, one can similarly prove it for pseudogroup homomorphisms into abstract groups, as it is stated in [12].)

Corollary 116. *Let S be a connected, simply connected topological group, V be a connected symmetric neighborhood of e in S and $f:V \to H$ be a local group homomorphism into a topological group H. Then there is a unique homomorphism $f':S \to H$ extending f.*

Proof. By Lemma 112, since S is connected and simply connected, the identity on S is a universal cover, and hence any universal cover of S by a connected group must be an isomorphism. We let $U := S$ in Theorem 113 (observing that $V^2 \subset S$), and obtain that

$\phi_{SV}: S_V \to S_S = S$ is a universal cover, and hence an isomorphism. The proof is now finished by Corollary 54. $\square$

Proposition 117. *If G is a locally defined group then G is simply connected.*

Proof. Let $h: G \to H$ be a homomorphism and $p: K \to H$ be a cover. Then by Lemma 115, for any sufficiently small symmetric neighborhood W of e in G there is a homomorphism $\psi: G_W \to K$ such that $p \circ \psi = h \circ \phi_{GW}$. By Proposition 61 we could choose W so that ϕ_{GW} is an isomorphism, and the proof is complete. $\square$

We can combine the previous two results to obtain:

Corollary 118. *A connected, locally connected group is simply connected if and only if it is locally defined.*

Theorem 119. *Let G be connected and locally connected. For any connected group $\overline{G}$ and cover $p: \overline{G} \to G$, the following are equivalent*:

(1) *$\overline{G}$ is locally defined.*
(2) *$\overline{G}$ is simply connected.*
(3) *p is simply connected.*
(4) *p is a universal cover.*

Moreover, such a connected group $\overline{G}$ and cover p exist (*and are unique up to isomorphism*) *if and only if G is semilocally simply connected.*

Proof. By the previous corollary and Lemma 112 we need only prove (4) $\Rightarrow$ (3). However, this implication follows from an easy pull-back argument (see [12, Proposition A2.4]). Existence follows from Theorem 113 and uniqueness is a standard argument. Conversely, if there is a simply connected cover $p: \overline{G} \to G$ and U, V are neighborhoods of e in $\overline{G}, G$, respectively such that the restriction of p to U is a homeomorphism onto V, then it follows immediately from the definitions that the inclusion of V into G is simply connected. $\square$

Note that the proofs of the above results require only basic results about Schreier groups.

The theory of Chevalley (C) is essentially contained in the above theorem. In fact, using a pull-back argument one can see immediately that Chevalley's definition of "locally simply connected" implies semilocal simple connectivity.

We will now show how to obtain (P) relatively simply from this theorem. If G is connected, locally arcwise connected, and semilocally simply connected in the traditional sense (hence in the current sense), Theorem 113 implies the existence of a (unique) universal cover $\phi: \overline{G} \to G$, where $\overline{G}$ is connected and locally defined. We now show that $\overline{G}$ is simply connected in the sense of (P). Since $\overline{G}$ is locally defined, by Proposition 61 we can find an arbitrarily small symmetric neighborhood V of e in $\overline{G}$ such that $\phi_{\overline{G}V}: \overline{G}_V \to \overline{G}$ is an isomorphism. In other words, every V-loop in $\overline{G}$ is V-homotopic to the trivial loop. Since $\overline{G}$ is locally arcwise connected and (traditionally) semilocally simply connected, we can further choose V so that every three points in V can be joined by curves to form a loop that is null-homotopic in $\overline{G}$. Now any loop α at e has a partition that is a V-loop. By the way V was chosen, any V-related V-loop α' can be filled in to a curve that is homotopic to α. Therefore any V-homotopy of α to the trivial loop can be filled in to an actual homotopy.

Theorem 120. *If G is connected and has a connected open neighborhood S of e whose inclusion into G is simply connected, then G has a connected universal cover.*

Proof. Consider the product $\prod G_U$ with the topology defined by the sets W_V defined below, where U is a symmetric neighborhood of e in G. Define a subgroup Y of $\prod G_U$ by $Y := \{(x_U)\colon \phi_{GU}(x_U) = \phi_{GV}(x_V)$ for all $U, V\}$, and let $\pi : Y_e \to G$ denote the restriction of the natural projection to Y_e. We will show that π is the desired universal cover.

Let U be an arbitrary symmetric neighborhood of e in G. Since G is connected, $\phi_{GU} : G_U \to G$ is a cover, and so by definition there exists a continuous function $f_U : S \to G_U$ such that $\phi_{GU} \circ f_U = i$, the identity on S. Since ϕ_{GU} is a local homeomorphism, f_U is open. Furthermore, $\phi_{GU} \circ f_U = i$ implies that f_U is one-to-one, hence a homeomorphism onto its image. Since S is connected, f_U is the unique such continuous function. For any symmetric neighborhood V of e, let

$$W_V := \bigl\{(x_U)\colon x_U \in f_U(V)\bigr\} = \prod f_U(V) \subset \prod G_U.$$

Since f_U is open, the sets W_V form a basis for some topology. Consider the continuous function $f : S \to Y$ given by $f(v) = (f_U(v))$; then $\pi \circ f$ is the identity on S, so f is one-to-one. For any symmetric neighborhood V of e in S, $f(V) = W_V \cap Y$. Therefore f is a homeomorphism. Since $W := f(U)$ is connected and open, $W \subset Y_e$ and generates Y_e. Additionally, the restriction of π to W is a homeomorphism, so $\pi : Y_e \to G$ is a cover.

Finally, suppose that $p : H \to G$ is a cover. By Lemma 115 there is a symmetric neighborhood U of e in G and a homomorphism $\psi : G_U \to H$ such that $\phi_{GU} = p \circ \psi$. To complete the proof we can let $s := \psi \circ \pi_U$, where $\pi_U : Y_e \to G_U$ is the restriction of the natural projection. □

Even if S is not connected, the above proof can be easily modified to show that the group Y is an (as a rule not connected!) universal cover of G. The only essential change is that, since the functions f_U may not be unique, we need to use the Axiom of Choice to assign one such function to each U. We obtain:

Theorem 121. *If G is connected and has a neighborhood S of e whose inclusion into G is simply connected, then G has a universal cover.*

Remark 122. The same "box product" construction can be used, together with the Axiom of Choice, to give an easy proof of the existence of universal covers even in the case of topological spaces.

Theorem 123. *Suppose G is a topological group with a connected universal cover. Then*

(1) *For any sufficiently small symmetric neighborhood U of e, the restriction of ϕ_{GU} to $(G_U)_e$ is a universal cover.*
(2) *$\widetilde{G}$ is connected and the natural homomorphism $\phi : \widetilde{G} \to G$ is a universal cover.*

Proof. Let $\eta : \overline{G} \to G$ be a universal cover, with $\overline{G}$ connected. Then G is connected. By Lemma 115 for any sufficiently small symmetric neighborhood U of G there is a unique homomorphism $\psi_U : G_U \to \overline{G}$ such that $\eta \circ \psi_U = \phi_{GU}$. According to Lemma 114, the restriction of ψ_U to $(G_U)_e$ is an isomorphism onto $\overline{G}$, and the first statement follows. Also from Lemma 114, $(G_U)_e$ is open in G_U. If $\overline{\phi}_{UV} : (G_V)_e \to (G_U)_e$ denotes the restriction of ϕ_{UV} then since $(G_V)_e$ is open in G_V, $\overline{\phi}_{UV}$ is open and hence surjective onto $(G_U)_e$. But then $\overline{\phi}_{UV}$ is a cover, hence an isomorphism (since by uniqueness, $\psi_U \circ \overline{\phi}_{UV} = \psi_V$ is an isomorphism on $(G_V)_e$). In particular, if $G' := \varprojlim((G_U)_e, \overline{\phi}_{UV}) \subset \widetilde{G}$ then the restriction of ϕ_U to G' is an isomorphism. Therefore G' is connected. If for any U we let $U' := (G_U)_e \cap U$ then U' is open in U (considered as a subset of G_U and G). But $\phi_{UU'} : G_{U'} \to G_U$ is surjective onto the subgroup of G_U that is generated by $U' \subset (G_U)_e$, so $\phi_{UU'}$ is surjective onto $(G_U)_e$. It follows from the definition of inverse limit that $G' = \widetilde{G}$ and the proof is complete. □

Corollary 124. *If $\psi : G \to H$ is a universal cover and G is connected then $\phi : \widetilde{G} \to G$ is an isomorphism.*

Proposition 125. *Let G be a topological group. Then the natural homomorphism $\phi : \widetilde{G} \to G$ is simply connected.*

Proof. Let $\psi : K \to H$ be a cover of topological groups and let $f : G \to H$ be a homomorphism. From Lemma 115, for some symmetric neighborhood W of e in G there exists a homomorphism $\eta : G_W \to K$ such that $\psi \circ \eta = f \circ \phi_{GW}$. Then $\eta \circ \phi_W : \widetilde{G} \to K$ is the needed continuous function. □

Theorem 126. *Suppose G is a topological group. For any connected group $\overline{G}$ and cover $p : \overline{G} \to G$, the following are equivalent*:

(1) *$\overline{G}$ is simply connected.*
(2) *p is simply connected.*
(3) *p is a universal cover.*

Moreover such a connected group $\overline{G}$ and cover p exist (*and are unique up to isomorphism*) *if G is connected and semilocally simply connected.*

Proof. From Lemma 112 we need only show (3) implies (1) to finish the equivalence. If p is a universal cover then $\phi : \widetilde{\overline{G}} \to \overline{G}$ is an isomorphism by Corollary 124, and $\overline{G}$ is simply connected by Proposition 125. Existence was proved in Theorem 120. □

Remark 127. Using a version of Theorem 120 for topological spaces we could strengthen the above theorem so that $\overline{G}$ is simply connected in the purely topological sense of Definition 108, but we do not see any advantage in doing so.

8. Special cases

Proposition 128. *Let H be a dense subgroup of a topological group G, U be a symmetric neighborhood of e in G, $U' = U \cap H$, and $i : H \to G$ denote the inclusion. Then*

(1) *the homomorphism $i' : H_{U'} \to G_U$ from Proposition* 53 *is an isomorphism onto its image, which is dense in G_U,*
(2) *G_U is locally generated if and only if $H_{U'}$ is locally generated, and*
(3) *if G_U is locally generated, $i'(H_{U'}) = \phi_{GU}^{-1}(H)$ and $i'(\ker \phi_{HU'}) = \ker \phi_{GU}$.*

Proof. Since U' is dense in U, G_U is generated by U and $H_{U'}$ is generated by U', and $i(U') = U'$, $i'(H_{U'})$ is dense in G_U. By the uniqueness of Proposition 53, the homomorphism i' must be the one that maps the U'-equivalence class of a U'-chain in H to the U-equivalence class of the same chain (which is a U-chain) in G. That is,

$$i'\big([c]_{U'}\big) = [c]_U.$$

We will first show that i' is open onto its image. To do this it suffices to prove that $i'(H_{U'}) \cap U = U'$. For then, $i'(U') = i(U') = U'$ is open in $i'(H_{U'})$. Suppose c is a U'-chain in H to some $x \in H$ and $i'([c]_{U'}) = [c]_U \in U$. Then $\phi_{GU}([c]_U) = x \in U'$. Since ϕ_{GU} restricted to U is the identity, it follows that then $[c]_U \in U'$.

To prove i' is injective and finish the proof of the first statement, we need some notation. By a U'-chain in G we mean a U-chain $\{x_0, x_1, \ldots, x_n\}$ such that $x_j \in H$ for all j. By a U'-homotopy in G we mean a U-homotopy $c_1, \ldots, c_n$ consisting of U'-chains. That i' is injective will follow if we can show: If $c_1, c_2, \ldots, c_{n-1}, c_n$ is a U-homotopy of U-chains to an element $x \in H$, such that c_1 and c_n are U'-chains, there exists a U'-homotopy $c_1, c_2', \ldots, c_{n-1}', c_n$. The proof is by induction on n. If $n = 2$, c_1, c_2 is already a U'-homotopy and there is nothing to prove. Suppose the result holds for some $n - 1 \geqslant 2$. Let $c_k = \{x_{k0}, \ldots, x_{km_k}\}$. Then since c_1 and c_2 are U-related, there is at most one $x_{2i} = p \notin U'$. If there is no such x_{2i} we do not change c_2; it is already a U'-chain and the proof is complete by the induction hypothesis. If some $x_{2i} = p \notin U'$, then since U' is dense in U, we can choose a $p' \in U'$ with the following property: for all k such that p is present in a chain c_k, i.e., $p = x_{kj}$ for some k, j with $2 \leqslant k < n$, if we replace $x_{kj} = p$ by p', the new chain $\widetilde{c}_k$ is still a U-chain. Having made all such replacements, we have a new sequence $c_1, \widetilde{c}_2, \ldots, \widetilde{c}_k, \ldots, c_n$ of U-chains, where $\widetilde{c}_2$ is now a U'-chain U'-related to c_1. If we can show that $\widetilde{c}_2, \ldots, \widetilde{c}_k, \ldots, c_n$ is still a U-homotopy, then the proof will be complete by the induction hypothesis. Consider two adjacent chains c_{j-1}, c_j where $3 \leqslant j \leqslant n$. The only difference between c_j and $\widetilde{c}_j$ is that every occurrence of p has been replaced by p'; the same is true of c_{j-1} and $\widetilde{c}_{j-1}$. Suppose, for example, c_j is obtained from c_{j-1} by adding a new point q. Then if $p \neq q$, $\widetilde{c}_j$ is still obtained from $\widetilde{c}_{j-1}$ by adding the point q, and so $\widetilde{c}_{j-1}$ and $\widetilde{c}_j$ are still U-related. Likewise $\widetilde{c}_{j-1}$ and $\widetilde{c}_j$ are still U-related if $q = p$; we are simply adding p' instead of p. A similar proof shows $\widetilde{c}_{j-1}$ and $\widetilde{c}_j$ are still U-related if $\widetilde{c}_j$ is obtained from $\widetilde{c}_{j-1}$ by removing a point. Therefore $\widetilde{c}_2, \ldots, \widetilde{c}_k, \ldots, c_n$ is still a U-homotopy; the proof of that i' is injective, and hence an isomorphism onto its image, is now complete.

Part (2) is an immediate consequence of Proposition 25 and part (1).

To prove part (3), suppose that G_U is locally generated. From Proposition 60 we have:

$$\phi_{GU} \circ i' = i \circ \phi_{HU'}. \tag{7}$$

This relation implies that $i'(H_{U'}) \subset \phi_{GU}^{-1}(H)$, so the latter group is dense in G_U. Therefore, by Proposition 25, $\phi_{GU}^{-1}(H)$ is locally generated. If, as usual, we consider U as a subset of G_U, then $U' = \phi_{GU}^{-1}(H) \cap U$ is open in $\phi_{GU}^{-1}(H)$. It follows that i' is open and therefore by Lemma 27; $i'(H_{HU}) = \phi_{GU}^{-1}(H)$. Finally, note that $\ker \phi_U \subset \phi_{GU}^{-1}(H)$. Since i' is surjective onto $\phi_{GU}^{-1}(H)$ and i is injective, we have from the relation (7),

$$i'(\ker \phi_{HU'}) = i'\bigl(\ker(i \circ \phi_{HU'})\bigr) = i'\bigl(\ker(\phi_{GU} \circ i')\bigr) = \ker \phi_{GU}. \qquad \square$$

Corollary 129. *A dense subgroup H of a topological group G is locally defined if and only if G is locally defined.*

Proof. Consider the correspondence $U' \leftrightarrow U$ of symmetric neighborhoods of e in H and G given by Proposition 128. Proposition 25 implies that H is locally generated if and only if G is locally generated. If both are locally generated, then the homomorphisms $\phi_{HU'}$ and ϕ_{GU} are both surjective. Since G (respectively H) is locally defined, by Proposition 61 we can choose U (respectively U') to be locally generated, and hence by Proposition 128(2), U' (respectively U) is locally generated. It follows from Proposition 128(3) that $\phi_{HU'}$ is injective if and only if ϕ_{GU} is injective. The proof of the corollary is now complete by Proposition 61. $\square$

Proof of Theorem 15. There is a one-to-one correspondence $U \leftrightarrow U' := U \cap H$, where U (respectively U') is a symmetric neighborhood of e in G (respectively H); moreover there is a one-to-one correspondence $\{U_\alpha\} \leftrightarrow \{U'_\alpha\}$ between bases of symmetric open sets of the topology at e for G and H. Proposition 128 implies that if every U'_α (respectively U_α) is locally generated then every U_α (respectively U'_α) is locally generated. In the metrizable case, we now see G is coverable if and only if H is coverable, by Theorem 92 (H is metrizable if and only if G is). In general, if H is coverable then since we are given that $\phi : \widetilde{G} \to G$ is surjective, we have by Theorem 90 that G is also coverable.

Now suppose there exist fixed bases $\{U_\alpha\}$ and $\{U'_\alpha\}$, with U_α and U'_α all locally generated, $U_{\alpha_0} = G$ and $U'_{\alpha_0} = H$. (Such bases can be found if either H or G is assumed coverable, by Theorem 90 and Proposition 128). Let $i'_\alpha : H_\alpha := H_{U'_\alpha} \to G_{U_\alpha} := G_\alpha$ denote the homomorphisms given by Proposition 128(3), which are all isomorphisms onto a dense subgroup of G_α, mapping $\ker \psi_{0\alpha}$ isomorphically onto $\ker \phi_{0\alpha}$, where $\phi_{\beta\alpha} : G_\alpha \to G_\beta$ and $\psi_{\beta\alpha} : H_\alpha \to H_\beta$ are the natural homomorphisms. Using the relation (7) and the uniqueness of the various homomorphisms we can easily obtain the relation

$$i'_\beta \circ \psi_{\beta\alpha} = \phi_{\beta\alpha} \circ i'_\alpha \tag{8}$$

whenever $\beta \leqslant \alpha$. Using this commutativity relation we can now make the following identifications: Identify the groups H_α with $i'_\alpha(H_\alpha)$. Identify the natural homomorphisms $\psi_{\beta\alpha} : H_\alpha \to H_\beta$ with the restrictions to $i'(H_\alpha)$ of $\phi_{\beta\alpha}$, and identify $\ker \psi_{0\alpha}$ with $\ker \phi_{0\alpha}$.

Hence $\widetilde{H} = \varprojlim(H_\alpha, \psi_{\beta\alpha})$ is naturally identified with a subgroup of $\widetilde{G} = \varprojlim(G_\alpha, \phi_{\beta\alpha})$, the natural homomorphisms $\psi_\beta : \widetilde{H} \to H_\beta$ are naturally identified with the restrictions of $\phi_\beta : \widetilde{G} \to G_\beta$ to $\widetilde{H}$, and $\tilde{i}$ is naturally identified with the inclusion of $\widetilde{H}$ into $\widetilde{G}$ (so is an isomorphism onto its image). $\widetilde{H}$ is dense in $\widetilde{G}$, since each H_α is dense in G_α.

We now return to the proof of the first statement of the theorem, if G is coverable. Then the homomorphism $\phi : \widetilde{G} \to G$ is surjective. For any $x \in H$ there exists a $y \in \widetilde{G}$ such that $\phi(y) = x$. Let $y_\alpha := \phi_\alpha(y)$. Since $\psi_{\alpha_0\alpha}$ is surjective, there exists some $y'_\alpha \in H_\alpha$ such that $x = \psi_{\alpha_0\alpha}(y'_\alpha) = \phi_{\alpha_0\alpha}(y_\alpha)$. Therefore $y_\alpha^{-1} y'_\alpha \in \ker \phi_{\alpha_0\alpha} = \ker \psi_{\alpha_0\alpha} \subset H_\alpha$, and $y_\alpha \in H_\alpha$. Since this is true for every α, $y \in \widetilde{H}$. According to our previous observations, the proof of the theorem is now complete, except for one final observation:

$$\pi_1(G) = \ker \phi = \varprojlim(\ker \phi_{\alpha_0\alpha}, \phi_{\beta\alpha}) = \varprojlim(\ker \psi_{\alpha_0\alpha}, \psi_{\beta\alpha}) = \ker \psi = \pi_1(H),$$

and so i_*, which is the restriction of the inclusion $\tilde{i}$ to $\ker \psi$, is an isomorphism between $\pi_1(H)$ and $\pi_1(G)$. □

Example 130. For this example we will construct a torsion-free totally disconnected group with non-trivial fundamental group. Let $\mathbb{R}$ be the reals with its usual topology, $\mathbb{Q}$ be the rationals with the induced topology, $\mathbb{Z}$ be the integers. Let $\Gamma = \sqrt{2}\mathbb{Z}$. Let $\pi : \mathbb{R} \to \mathbb{R}/\sqrt{2}\mathbb{Z}$ be the quotient epimomorphism. Then $H := \pi(\mathbb{Q})$ is a dense subgroup of the circle $\mathbb{R}/\sqrt{2}\mathbb{Z}$, and so by Theorem 15 is coverable, and $\pi_1(H)$ is isomorphic to $\mathbb{Z}$.

The next result is an immediate consequence of Corollary 118:

Proposition 131. *Every real topological vector space V is locally defined.*

We now consider a family of examples that are connected with measure and ergodic theory. Let $(\mathcal{S}, \Sigma, \mu)$ be a measure space, where Σ is a σ-algebra of subsets of $\mathcal{S}$, and μ is the measure. We let Σ' be the set of equivalence classes of elements of Σ of finite measure, where two elements are equivalent if they differ by a set of measure 0. The product operation is induced by the operation $AB = A \triangle B := (A \backslash B) \cup (B \backslash A)$ on Σ, and the identity element is the equivalence class of $\emptyset$. Where no confusion will result, we will not distinguish notationally between a set and its equivalence class. Note that Σ' is an Abelian group such that every non-empty element of Σ' has order 2. We define the usual invariant metric on Σ' by $d(A, B) = \mu(AB)$, and let Σ' have the topology defined by this metric. Note that if A and B overlap in a set of measure 0, then $\mu(AB) = \mu(A) + \mu(B)$. This construction from analysis was considered by van Mill [17] from the standpoint of topological groups. In [17] he made the incorrect claim that, for $\mathcal{S} = [0, 1]$ with Lebesgue measure, this group has no homeomorphisms besides translations. In fact, by a theorem of Caratheodory, there is a measure preserving isomorphism between the σ-algebra of Lebesgue measurable subsets of $[0, 1]$ and the σ-algebra of any non-atomic measure space of finite measure normalized to 1 (i.e., a *Lebesgue space*). Therefore, if $(\mathcal{S}, \Sigma, \mu)$ is any Lebesgue space then any measure preserving automorphism of Σ corresponds to an isometry (hence homeomorphism) of the van Mill group (Σ', d), that fixes the identity and therefore cannot be a nontrivial translation. There are plenty of such nontrivial automorphisms supplied by ergodic theory (e.g., from the geodesic flow on a compact Riemannian manifold and Bernoulli shifts), see, e.g., [20], for more details. Conversely, every isometry of Σ' preserving the identity corresponds to a measure preserving automorphism of Σ and therefore an (outer) automorphism of the group Σ'. If the group Σ' is viewed as the quotient group $L^1(\mathcal{S},\mathbb{Z})/L^1(\mathcal{S},2\mathbb{Z})$ then Σ' is seen to be connected and arcwise connected. However, this topological fact can be strengthened to a geometric one. We summarize its properties:

Proposition 132. *If $\mathcal{S}$ is $[0,1]$ or the set of all real numbers, Σ is the σ-algebra of Borel sets and μ is Lebesgue measure, then the topological group Σ' defined above is Abelian, complete, metrizable, connected, locally arcwise connected, and contractible* (*hence locally defined*), *and every nontrivial element has order* 2. *The metric d defined above is a complete invariant inner metric such that every pair of points can be joined by a minimal geodesic.*

Proof. The last statement follows from the fact that if $\mu(A)>0$ then A can be written as the union of two sets of measure $\mu(A)/2$. In other words, between every two points of Σ' there is a midpoint. The metric d is well-known to be complete, and so such midpoints can be used to construct a curve γ joining any two points A,B, whose length is equal to $d(A,B)$. Such a curve γ is by definition a minimal geodesic. The existence of minimal geodesics joining points proves that Σ' is connected and locally arcwise connected. We have seen that Σ' is Abelian and every nontrivial element has order 2, so we need only show Σ' is contractible.

For any Borel set A and $t\in[0,1]$, let $tA=\{ta\colon a\in A\}$. We claim that the mapping $H:\Sigma'\times[0,1]\to\Sigma'$ given by $H(A,t):=H_t(A)=tA$ is continuous. Since H_0 maps Σ' to (the equivalence class of) $\emptyset$, this will complete the proof. First note that for any interval $I=[a,b]$, $tI=[ta,tb]$, and so $\mu(tI)=t\mu(I)$. From standard arguments in elementary analysis we obtain that for any $A\in\Sigma'$, $\mu(tA)=t\mu(A)$. Note also that $(tA)(tB)=t(AB)$, so $\mu(tA,tB)=t\mu(A,B)$. Now fix $t,\varepsilon>0$ and $A\in\Sigma'$, and suppose that $\mu(A,B)<\frac{1}{2}\varepsilon$. Then for any $s\in[0,1]$,

$$\begin{aligned}\mu\big(H(A,t),H(B,s)\big) &= \mu(tA,sB)\leqslant\mu(tA,sA)+\mu(sA,sB)\\ &\leqslant \mu(tA,sA)+\tfrac{1}{2}s\varepsilon\leqslant\mu(tA,sA)+\tfrac{1}{2}\varepsilon.\end{aligned}$$

Therefore to complete the proof we need to show that if s is sufficiently close to t then $\mu(tA,sA)$ is arbitrarily small. Certainly this is true if A is an interval, or a finite union of intervals. By elementary analysis, there exists a set C such that C is a finite union of intervals and $\mu(A,C)\leqslant\frac{1}{6}\varepsilon$, and by the previous sentence, if s is sufficiently close to t, $\mu(sC,tC)\leqslant\frac{1}{6}\varepsilon$. For such s we have

$$\mu(sA,tA)\leqslant\mu(sA,sC)+\mu(sC,tC)+\mu(tC,tA)\leqslant\tfrac{1}{2}\varepsilon.\qquad\square$$

Example 133. The group $SU(2)$, which is well-known to be simply connected, contains non-trivial closed 1-parameter subgroups. This shows that a closed, coverable subgroup of a locally defined group need not be locally defined. If $\phi : S \to SU(2)$ denotes the inclusion of a circle subgroup S into $SU(2)$, then ψ is injective, but the induced map $\widetilde{\psi} : \mathbb{R} \to SU(2)$ is not.

Finally, we mention that the "Abelian weak Lie" groups considered in [11] are also coverable, by [11, Theorem 2.8].

9. Open problems

Unless otherwise stated, all problems concern the category $\mathcal{T}$ of all topological groups.

Problem 134. Is every closed prodiscrete subgroup of a coverable (or more generally locally generated) group central?

Problem 135. Is the composition of covers between locally generated (or more generally topological) groups again a cover? That is, do locally generated groups, with covers as morphisms, constitute a category? If so, is a locally generated group G coverable if and only if there is a universal covering group for G in the categorial sense of Theorem 7?

Problem 136. Is $\widetilde{G}$ always locally defined, for any topological group G?

Problem 137. If $\phi : \widetilde{G} \to G$ is an isomorphism, is G locally defined?

Problem 138. If G is simply connected and locally generated, is G locally defined?

Problem 139. If G is locally generated, does the natural homomorphism $\phi : \widetilde{G} \to G$ have dense image in G?

Problem 140. Is the inverse limit of locally generated groups locally generated?

Problem 141. Are there examples of topological groups G such that G^l is properly contained in the quasicomponent of G (see Section 2)?

Problem 142. Does every topological group have a "coverable component", i.e., a largest coverable subgroup?

Problem 143. In what other categories does the notion of locally defined group give rise to a covering group theory? For example, we show in [3] that there are no non-trivial locally defined groups in the category $\mathcal{K}$ of compact, connected groups, so other methods (i.e., duality) must be used to obtain a covering group theory in $\mathcal{K}$.

Cantor gave a definition of connectedness for metric spaces that is more general than the usual one (see [25]). In the case of a compatible left-invariant metric on a topological group G, Cantor's definition is equivalent to the first half of the following:

Definition 144. A subset A of a topological group G is called Cantor connected (or C-connected) if for all points x, y of A and for every symmetric open neighborhood U of e in G there is a U-chain in A joining x and y. The group G is called locally C-connected if there is a basis for the topology of G at e consisting of C-connected open sets.

Proposition 20(3) states that G is C-connected if and only if G is locally generated. It is not hard to prove that a metrizable group that is Cantor connected and locally Cantor connected must be coverable.

Problem 145. Is a coverable topological group Cantor connected and locally Cantor connected?

Problem 146. Is the universal covering group of a metrizable, coverable, connected (respectively locally connected, locally arcwise connected) group connected (respectively locally connected, locally arcwise connected)?

We hope to address some topological questions in a later paper. In particular, we would like to relate our construction of the universal cover to the usual construction involving homotopy classes of curves in the case when G is locally generated and locally arcwise connected (which together easily imply connectedness). We conjecture that in this case $\widetilde{G}$ is again connected and locally arcwise connected.

Problem 147. Do there exist complete groups that are connected and coverable but not locally connected?

Definition 148. A connected, locally arcwise connected group G is called almost simply connected if for any loop γ based at e and open neighborhood U of e, γ is base-point homotopic to a loop γ' based at e contained in U.

Problem 149. Is an almost simply connected cover of a coverable group isomorphic to its universal covering group?

Problem 150. Is there an almost simply connected group that is not simply connected in the sense that every loop is null-homotopic?

Such a group would not be "homotopically Hausdorff". By definition, a topological space is homotopically Hausdorff at a point p if every loop based at p that can be homotoped into arbitrarily small neighborhoods of p, is null-homotopic (cf. [7,10]). In particular, the traditional fundamental group at p in such a group, consisting of homotopy classes of loops based at p, with the compact-open topology, would not be a Hausdorff topological space. In contrast, the fundamental group defined in this paper (with its natural prodiscrete topology) is always Hausdorff.

Problem 151. Does there exist a (metrizable) connected, locally arcwise connected group that is simply connected but not locally simply connected in the sense that every neighborhood of e contains a simply connected neighborhood of e?

Problem 152. Is the fundamental group defined in this paper a topological invariant? That is, if two coverable groups are homeomorphic, must their fundamental groups be isomorphic?

Problem 153. Is there an analog of our group $\pi_1(G)$ for higher homotopy groups?

In this connection it is useful to note that the description of G_U in terms of U-chains and their U-homotopy classes, as well as the definition of $\widetilde{G}$ as an inverse limit of the groups G_U, resemble the classical description of Vietoris homology groups, cf. [5].

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